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Par

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Un calcul paracontrôlé pour les EDP stochastiques singulières sur les variétés

Vers l'infini et au-delà

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Contents

Aperçu de la thèse

Cette thèse porte sur la résolution d'équations aux dérivées partielles stochastiques singulières (EDPSS) sur des variétés. Pour cela, on construit un calcul paracontrôlé d'ordre supérieur basé sur le semi-groupe de la chaleur. L'étude de chaque équation a ensuite lieu en deux étapes : la formulation analytique du problème avec les outils du calcul paracontrôlé et la construction d'un certain nombre de processus aléatoires singuliers par une procédure de renormalisation.

Les objets modélisant un bruit aléatoire sont par nature très irréguliers localement. Cela rend l'étude des EDP associées délicate lorsque des produits entre deux objets irréguliers apparaissent et les équations ne peuvent alors pas être résolues dans les espaces classiques d'analyse. En se basant sur la même approche que les chemins rugueux et les chemins contrôlés pour les équations différentielles stochastiques, il est possible de construire des sous-espaces aléatoires à l'aide de méthodes probabilistes dans lesquels les produits ne sont plus singuliers. Cela permet d'étudier de nouveaux modèles de processus aléatoires continus, de mieux comprendre les limites d'échelles d'un certain nombre de systèmes dynamiques discrets aléatoires ou encore de construire des modèles de théorie quantique des champs. Le chapitre 1 porte sur la construction du calcul paracontrôlé à partir du semi-groupe de la chaleur qui permet de mettre en pratique cette idée. Dans les chapitres 2 à 6, différents exemples d'EDPSS paraboliques, elliptiques et dispersives sont étudiés. Enfin, le chapitre 7 porte sur deux exemples de modèles aléatoires étudiés grâce à des opérateurs stochastiques singuliers construits à l'aide du calcul paracontrôlé.

L'étude de ces objets stochastiques irréguliers est alors difficile avec les outils habituels d'analyse et de nouvelles méthodes sont nécessaires. L'intégration stochastique a été développée en ce sens mais de nombreuses questions comme la résolution des équations KPZ ou du modèle Φ_d^4 restaient hors de portée des méthodes d'analyse ou de probabilités jusqu'à récemment. Depuis le début des années 2000, de nombreux progrès ont été fait avec en particulier l'introduction des structures de régularité et du calcul paracontrôlé qui permettent aujourd'hui d'étudier de nombreux problèmes jusque là inabordable. Cette thèse n'est qu'un petit exemple de la nature des nombreuses questions qu'on est alors à même de se poser.

Chapitre 1 : Construction du calcul paracontrôlé.

On construit les outils du calcul paracontrôlé d'ordre supérieur sur une variété compacte à l'aide du semi-groupe de la chaleur dans un cadre espace, en particulier les deux paraproduits P et $\widetilde{\mathsf{P}}$. Le premier permet de décomposer un produit comme

$$ab = \mathsf{P}_a b + \mathsf{P}_b a + \Pi(a,b)$$

où $\mathsf{P}_a b$ et $\mathsf{P}_b a$ sont bien définis pour toutes distributions alors que le terme résonant $\Pi(a,b)$ contient l'éventuelle singularité. Le deuxième paraproduit $\widetilde{\mathsf{P}}$ est entrelacé avec P par une relation du type

$$D \circ \widetilde{\mathsf{P}} = \mathsf{P} \circ D$$

avec D un opérateur différentiel et permet ainsi d'exprimer les formulations faibles d'EDP impliquant un produit décrit par P. Le paraproduit P est construit à partir du semi-groupe de la chaleur associé à une famille d'opérateur différentiel du premier ordre $(V_i)_{1\le i\le d}$ vérifiant de bonnes hypothèses, D peut être alors être n'importe quel opérateur construit à partir de cette famille. Dans le cas du calcul paracontrôlé espace-temps, il est aussi possible de considérer l'opérateur temporel ∂_t. Dans les applications, on considère $D = -\Delta$ ou $D = \partial_t - \Delta$. Pour réussir à contourner le problème des produits singuliers, on obtient des estimations de continuité sur des correcteurs du type

$$\mathsf{C}(a_1, a_2, b) := \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - a_1 \mathsf{\Pi}(a_2, b)$$

ainsi que pour ses versions rafinées et itérées. D'autres opérateurs sont aussi nécessaires pour étudier des termes bien définis mais qui ne sont pas dans une forme adaptée comme par exemple le commutateur

$$\mathsf{D}(a_1, a_2, b) := \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - \mathsf{P}_{a_1} \mathsf{\Pi}(a_2, b)$$

ou encore l'opérateur de fusion

$$\mathsf{R}(a, b, c) := \mathsf{P}_a \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} c.$$

Chapitre 2 : EDP stochastiques singulières paraboliques semi-linéaires.

À l'aide du calcul paracontrôlé d'ordre supérieur, il est possible de résoudre une grande classe d'EDP stochastiques singulières paraboliques semi-linéaires comme par exemple l'équation (PAM) généralisée en dimension 3

$$\partial_t u - \Delta u = f(u)\xi$$

avec ξ un bruit blanc espace ou l'équation (KPZ) généralisée en dimension $1+1$

$$\partial_t u - \partial_x^2 u = f(u)\zeta + g(u)(\partial_x u)^2$$

avec ζ un bruit blanc espace-temps. Notre approche repose sur la notion de systèmes paracontrôlés $(u_a)_{a\in\mathscr{A}}$ étant donné un ensemble $\mathcal{T}$ de fonctions de référence dépendantes du bruit vérifiant un système triangulaire

$$u_a = \sum_{|a\tau|\le n\alpha} \widetilde{\mathsf{P}}_{u_{a\tau}} \tau + u_a^\sharp$$

où $|\cdot|$ correspond à la régularité Hölder des différentes fonctions. La singularité de l'équation apparait alors par la singularité de certains éléments de $\mathcal{T}$ qui doivent être construit à l'aide d'une procédure de renormalisation. Les équations sont ensuite résolues presque sûrement par un argument de point fixe sur l'espace des restes $(u_a^\sharp)_{a\in\mathscr{A}}$. On présente ici seulement la partie concernant la formulation du point fixe.

Chapitre 3 : EDP stochastiques singulières paraboliques quasi-linéaires.

- I. Bailleul and A. Mouzard, *Paracontrolled calculus for quasilinear singular PDEs*, arXiv:1912.09073, (2019).

On développe de nouveaux outils pour le calcul paracontrôlé d'ordre supérieur afin de résoudre la partie analytique de l'étude d'EDP singulières quasi-linéaires. En plus de l'étude de nouveaux correcteurs et commutateurs introduits dans ce but, on généralise la notion de systèmes paracontrôlés afin de pouvoir travailler avec un nombre infini de fonctions de références générées par une structure algébrique finie. On considère toute EDP quasi-linéaire

$$\partial_t u - d(u)\Delta u = f(u,\xi)$$

associée à une EDP semi-linéaire qui peut être traitée à l'aide le cadre du calcul paracontrôlé, ce qui inclut (gPAM) et (gKPZ). L'équation est reformulée comme

$$\partial_t u + Lu = f(u,\xi) + \varepsilon(u,\cdot)Lu + a_i(u,\cdot)V_i u$$

avec L un opérateur différentiel du second ordre dépendant de la condition initiale u_0 sous la forme de Hörmander afin d'utiliser le calcul paracontrôlé adapté. En introduisant de nouveaux correcteurs et commutateurs associés au produit mettant en jeu les opérateurs différentiels L et V_i, on est capable de formuler le point fixe associé à l'équation dans un espace de système paracontrôlé. À cause du terme de second ordre dans le membre de droite, on ne peut pas trouver un ensemble fini de fonctions de références $\mathcal{T}$ et il est nécessaire de considérer des entiers comme "décorations" sur l'ensemble fini donné par l'équation semi-linéaire. Le passage d'un espace algèbrique de dimension finie à infinie est aussi présent dans les autres approches développées pour ces équations, cela semble être l'effet quasi-linéaire.

Chapitre 4 : L'hamiltonien d'Anderson.

- A. Mouzard, *Weyl law for the Anderson Hamiltonian on a two-dimensional manifold*, arXiv:2009.03549, (2020).

On définit l'opérateur d'Anderson

$$H = -\Delta + \xi$$

sur une variété de dimension 2 à l'aide du calcul paracontrôlé d'ordre supérieur où ξ est un bruit blanc espace. À cause de la singularité de l'opérateur, la construction nécessite une procédure de renormalisation et on obtient un opérateur auto-adjoint avec un spectre discret $\big(\lambda(\Xi)\big)_{n\ge1}$. On peut calculer la régularité Hölder des fonctions propres et obtenir des bornes inférieures et supérieures pour ses valeurs propres de la forme

$$\lambda_n - m_\delta^2(\Xi) \le \lambda_n(\Xi) \le (1+\delta)\lambda_n + m_\delta^1(\Xi)$$

pour tout $\delta \in (0,1)$, des constantes explicites m_δ^1, m_δ^2 qui dépendent de l'enrichissement du bruit Ξ et $(\lambda_n)_{n\ge1}$ les valeurs propres du laplacien. En particulier, ces bornes impliquent une loi de type Weyl presque sûre pour H de la forme

$$\lim_{\lambda\to\infty} \lambda^{-1}\big|\{\lambda_n(\Xi)\le\lambda\}\big| = \frac{\mathrm{Vol(M)}}{4\pi}.$$

Chapitre 5 : Le laplacien magnétique aléatoire.

- L. Morin and A. Mouzard, *2D random magnetic Laplacian with white noise magnetic field*, arXiv:2101.05020, (2021).

On construit le laplacian magnetique aléatoire

$$H = (i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2$$

sur le tore de dimension 2 avec un potentiel magnétique $A = (A_1, A_2) \in \mathcal{C}^{\alpha-1} \times \mathcal{C}^{\alpha-1}$ aléatoire irrégulier où $\alpha < 1$ en utilisant le calcul paracontrôlé. Le potentiel est choisi de manière à ce que le champ magnétique associé soit le bruit blanc espace avec

$$A = \nabla^\perp \varphi \quad \text{où} \quad \varphi = \Delta^{-1}\xi.$$

Après une procèdure de renormalisaiton, on obtient un opérateur auto-adjoint avec un spectre discret. On obtient aussi des bornes inférieures et supérieures pour ses valeurs propres qui implique une loi de type Weyl presque sûre. En particulier, notre construction est un exemple d'opérateur de la forme

$$-\Delta + a_1 \cdot \nabla + a_2$$

avec a_1, a_2 des champs aléatoires plus irrégulier que ce qui est possible avec la théorie classique.

Chapitre 6 : EDP stochastiques singulières dispersives.

- A. Mouzard and I. Zachhuber, *Strichartz inequalities with white noise potential on compact surfaces*, arXiv:2104.07940, (2021).

On résout des EDP dispersives non-linéaires avec un bruit blanc multiplicatif à l'aide de la construction de l'hamiltonien d'Anderson qui permet par exemple d'interpréter l'équation de Schrödinger

$$i\partial_t u + \Delta u = u\xi + |u|^2 u$$

comme

$$i\partial_t u = Hu + |u|^2 u.$$

Les différentes propriétés de H permettent d'obtenir des solutions fortes et énergies pour cette équation ainsi que pour l'équation des ondes sur une surface compacte sans bord ou avec des conditions de Dirichlet au bord. Dans le cas déterministe, les inégalités de Strichartz de la forme

$$\|e^{it\Delta}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha}$$

permettent d'obtenir une théorie de solutions dans des espaces de Sobolev de faible régularité. On prouve de telles inégalités pour l'hamiltonien d'Anderson, c'est-à-dire de la forme

$$\|e^{itH}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha}$$

ce qui permet d'obtenir le caractère bien posé localement dans des espaces de Sobolev de faible régularité pour les équations de Schrödinger et des ondes avec une non-linéarité cubique et un bruit multiplicatif sur une surface compacte avec ou sans bord. En particulier, on obtient des bornes sur les normes L^q des fonctions propres de l'hamiltonien d'Anderson et ses projecteurs spectraux.

Chapitre 7 : Diffusions en milieu désordonné.

- A. Mouzard, *The continuum polymer measure with white noise potential on compact surfaces*, in preparation.
- A. Mouzard, *The Brox diffusion on a circle and its generator*, in preparation.

On étudie deux modèles de diffusions en milieu désordonné à l'aide d'opérateurs stochastiques singuliers. Le premier est celui de la mesure polymère avec potentiel bruit blanc ξ formellement décrite par

$$\mathbf{V}(\mathrm{d}X) = \frac{1}{Z_T} e^{-\frac{1}{2}\int_0^T \xi(X_s)\mathrm{d}s} \mathbf{W}(\mathrm{d}X)$$

où $\mathbf{W}$ est la mesure de Wiener sur $C([0,T],M)$. Puisque le bruit blanc est seulement une distribution, le terme $\xi(X_s)$ n'a pas de sens et le formalisme de Gibbs ne peut pas être utilisé pour construire un tel objet. Cela fait sens puisque dans ce cas, la mesure polymère est en fait singulière avec la mesure de Wiener et ne peut donc par admettre une densité par rapport à cette dernière. Notre construction est basée sur le semi-groupe intrinsèque de Feynman-Kac associé à l'hamiltonien d'Anderson et est reliée à la diffusion

$$\mathrm{d}X_t = \nabla(\log\Psi)\mathrm{d}t + \mathrm{d}B_t$$

où Ψ est l'état fondamental d'Anderson. Le second modèle est la diffusion de Brox formellement donnée par l'EDS

$$\mathrm{d}X_t = \xi(X_t)\mathrm{d}t + \mathrm{d}B_t$$

où la dérive est singulière et donné par un bruit blanc espace ξ en dimension un. Son générateur infinitésimal est formellement donné par

$$-\frac{1}{2}\Delta + \xi\cdot\nabla$$

qui peut être construit à l'aide du calcul paracontrolé basé sur le semi-groupe de la chaleur. En particulier, cela permet une nouvelle approche pour l'étude de la diffusion de Brox ainsi que sa construction sur le cercle où on ne peut utiliser la propriété d'auto-similarité. L'étude de ces deux modèles étant basée sur des travaux en cours, ce chapitre présentera seulement les idées générales.

Introduction

The natural continuous objects that appear in stochastic analysis are very rough as one already see with the Brownian motion. This made stochastic integration a real challenge to develop with motivation the resolution of stochastic differential equation (SDEs) to study new probabilist models and undestand the scaling limits of discrete random dynamical systems. The same is true for the resolution of PDEs involving complex operations with random noise distributions and a large number of questions such as the resolution of the KPZ or Φ^4_d equations remained out of reach of known methods until very recently. Since the early 2000's, major progess have been made, in particular with the introduction of regularity structures and paracontrolled calculus allowing the study of many problems inacessible until then. This thesis is only a small sample of the kind of questions one is now able to investigate.

What, why and how

Singular stochastic partial differential equations (SSPDEs) are PDEs involving rough stochastic fields as source terms or initial conditions where a singular operation appears. Since the stochastic term is not a function, one has to deal with a product of distributions which is ill-defined in most cases. Given a discrete space Λ, one of the most natural stochastic noise one can consider is a family $\{\xi(x); x \in \Lambda\}$ of independant and identically distributed centered Gaussian random variables. In a continuous setting, the analogue object ξ is called white noise and one can not expect it to belong to any classical function spaces because of its roughness; it is not a function but only a distribution. This stochastic object can appear in any classical problems of analysis such as parabolic PDEs like the multiplicative heat equation

$$\partial_t u - \Delta u = u\xi,$$

elliptic PDEs like the Laplace equation

$$\Delta u = u\xi,$$

hyperbolic PDEs like the multiplicative wave equation

$$\partial_t^2 u - \Delta u = u\xi$$

or dispersive PDEs like the multiplicative Schrödinger equation

$$i\partial_t u - \Delta u = u\xi.$$

In all these equations, the solutions u might not be regular enough for the product $u\xi$ to makes sense almost surely; this is the so-called singular SPDEs class. As for stochastic differential equations where a probabilistic argument is needed to go beyond the limit of analysis, solving singular SPDEs comes down to the use of probability to push back the limit of possible regularity in PDEs. There are also equations where the singular product does not directly involve the noise such as the Φ^4_d equation

$$\partial_t \phi - \Delta \phi = \xi - \phi^3 + \phi$$

or the Kardar–Parisi–Zhang (KPZ) equation

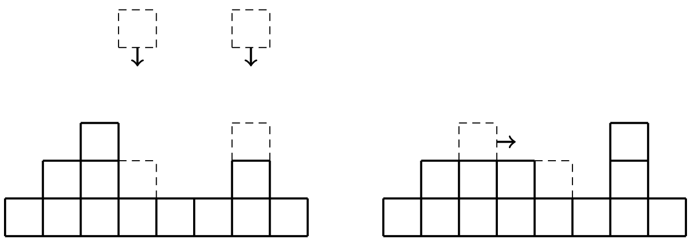
$$\partial_t u - \partial_x^2 u = \xi + (\partial_x u)^2$$

where the terms u^3 and $(\partial_x u)^2$ do not make sense for rough noise ξ. Since there is no natural meaning of what a solution should be at first sight, it is normal to ask why one should try to solve these equations, or at least keep faith to do so. It rapidly seems clear that there is no hope to find any classical solutions thus it is crucial to see the greater picture and keep in mind the problem behind the equation that one wants to solve. In general, two types of problems give rise to singular SPDEs.

– A large number of nonlinear microscopic random systems are described by singular SPDEs on a macroscopic level. One of the most famous one is the generalised KPZ equation

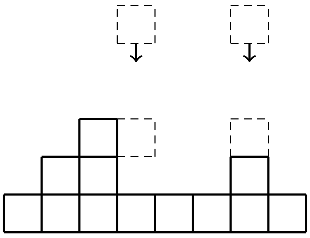
$$\partial_t u - \partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

with ζ a spacetime white noise which describes a whole range of interacting particle systems. For example, the case $f = g = 1$ corresponds to the weakly asymmetric simple exclusion process. An interesting family of discrete growth models to illustrate different stochastic processes associated to such SPDEs is the random deposition model. The simplest one is described as follows. One adds new particules at random time and choses uniformly a site in which the particule falls verticaly until it reaches the top of the column. Since there is no correlation between the different columns, the stochastic process can be explicitely computed. The interesting quantity to consider is the fluctuation of the interface line around its mean. In the correct continuous limit, this is given by a space-time white noise, Gaussian because of the Central Limit Theorem.



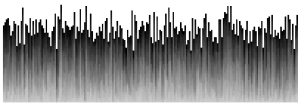
Random deposition *Surface relaxation*

There are different ways to induce a correlation between columns which yield different SPDEs in the continuous scaling limit. For example, one can induce a surface relaxation by allowing deposited particules to diffuse to an adjacent lower height. The final interface will be smoother than without relaxation and one can show that the continuous limit model is described by the stochastic heat equation with additive spacetime white noise often refered to as the Edwards-Wilkinson equation in this framework. Another example of model with correlation is the ballistic deposition model where a deposited particule sticks to the first edge against which it becomes incident.

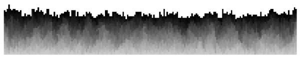


Ballistic deposition

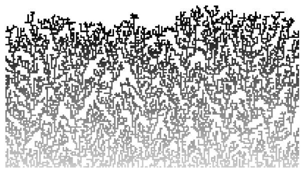
While the overall growth rate remains the same, the fluctuation of the interface line around its mean are very different and falls in the so-called KPZ universality class. These three models of surface growth appear in many natural phenomena such as coffee stain or ice deposition and exhibit very different behaviors.



Random deposition



Relaxed random deposition



Ballistic deposition

– Quantum Field Theory (QFT) is an attempt to unify the quantum theory of particule physics with the theory of relativity. While it takes its root in Dirac's theory of antiparticules, it had the major problem of having no concrete meaning : each computations yielded infinite values. It is only with the works of Bethe, Tomonaga, Dyson, Schwinger and Feynman that one could get meaningful computations after a so-called "renormalisation" procedure. It consisted in a sequence of operations performed for the theory to make sense, this was refered to as constuctive Quantum Field Theory (cQFT). The fact that the renormalisation procedures produced the most precise computations of the anomalous magnetic dipole moment gave the hint that the problem was not in QFT but in the actual construction of a QFT. Thus cQFT appeared as a very hard but interesting mathematical challenge, the construction of a measure on an infinite dimensional space having a density with respect to the "Lebesgue" measure satisfying a number of axioms. The problem of ill-definition already appears with the interpretation of the "Lebesgue" measure in infinite dimensions. The simplest example is the Φ^4_d theory on the torus $\mathbb{T}^d$ given by

$$\mu(\mathrm{d}\phi) = \frac{1}{Z} e^{-V(\phi)} \prod_{x \in \mathbb{T}^d} \mathrm{d}\phi(x)$$

where Z is a normalisation constant and

$$V(\phi) := \int_{\mathbb{T}^d} \Big(|\nabla \phi(x)|^2 + \frac{1}{2}\phi(x)^4 - \phi(x)^2 \Big) \mathrm{d}x.$$

A first step is to remark that

$$\exp\left(- \int_{\mathbb{T}^d} |\nabla \phi(x)|^2 \mathrm{d}x \right) = \exp\left(- \int_{\mathbb{T}^d} \phi(x)(-\Delta)\phi(x) \mathrm{d}x \right)$$

and to consider the measure ν defined as

$$\nu(\mathrm{d}\phi) = \exp\left(-\int_{\mathbb{T}^d} \phi(x)(-\Delta)\phi(x)\mathrm{d}x\right) \prod_{x\in\mathbb{T}^d} \mathrm{d}\phi(x).$$

Following Gaussian measures in finite dimensions, ν can be interpreted as the Gaussian measure with

$$\int \langle \phi, f\rangle\langle \phi, g\rangle \nu(\mathrm{d}\phi) = \frac{1}{2}\langle f, (-\Delta)^{-1}g\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mathbb{T}^d)$. Then the Φ_4^d measure is given by

$$\mu(\mathrm{d}\phi) = \frac{1}{Z}\exp\left(-\int_{\mathbb{T}^d}\left(\frac{1}{2}\phi(x)^4\mathrm{d}x - \phi(x)^2\right)\mathrm{d}x\right)\nu(\mathrm{d}\phi)$$

and this gives a well-defined measure in dimension $d = 1$. In dimension $d \geq 2$, the measure ν is only supported in distributions with $\nu(L^\infty) = 0$ hence the product ϕ^4 and ϕ^2 do not make sense. Considering the SPDE with μ as formal invariant measure yields the Φ_d^4 equation

$$\partial_t\phi - \Delta\phi = -\phi^3 + \phi + \xi$$

with ξ a spacetime white noise which is singular in dimension $d \geq 2$; this is the idea of stochastic quantization introduced by Parisi and Wu.

Now that "what" and "why" are clear, it remains to see the "how". A classic approach is to consider a regularisation of the noise $(\xi_\varepsilon)_{\varepsilon>0}$ which converges to ξ as ε goes to 0. If the regularisation is well-chosen, this yields a family of well-posed problems with associated solution $(u_\varepsilon)_{\varepsilon>0}$ and the question is now to describe the asymptotic behavior of u_ε as ε goes to 0. Since the equations are singular, one can not expect this family to converge but can only hope to get a description of its asymptotic behavior; this is where a renormalisation procedure appears.

Throughout this thesis, the reader should keep in mind our goal : we want to build an object independent of ε describing the asymptotic behavior of $(u_\varepsilon)_{\varepsilon>0}$. One should apply the simplest possible transformation to the equation in order to get a new family $(v_\varepsilon)_{\varepsilon>0}$ that converges to a distribution v; this is the renormalisation step. Hence the problem is to find this transformation, construct the space of possible limits and show the convergence. The convergence should rely on a probabilistic argument since the problems do not have natural meaning with pure analysis. In order to have a precise description of the singular product and construct this limiting object v, one needs adapted analysis tools; this is the role of regularity structures or paracontrolled calculus.

The presence of a rough stochastic source term already appears in the context of differential equation where one wants to add a Brownian motion. Such stochastic differential equations are often written under the form

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}B_t$$

and one has to give a meaning to "$\mathrm{d}B_t$" since the Brownian motion is almost surely not differentiable. From a pathwise point of view, B is almost surely of Hölder regularity $\frac{1}{2} - \kappa$ for any $\kappa > 0$ hence to solve the SDEs for almost every $\omega \in \Omega$, one has to have a theory that can deal with this regularity. Unfortunaly, this is just out of range of Young's integration (1936) where one can give a meaning to

$$\int_0^t f(Y_s)\mathrm{d}Y_s$$

for any path Y of Hölder regularity greater than $\frac{1}{2}$. The condition being sharp, the study of SDEs with Ito's (1944) and Stratonovich's (1966) theories of stochastic integration were developped. The addition of a probabilistic ingredient allows a definition of the integral for a class of stochastic processes against each other. In the late 90's, Lyons took back the path of stochastic integration from the work of Young and developped a pathwise approach to SDEs. The idea is to build a random function space from the Brownian motion in which one can make sense of the equation almost surely. This random function space is built using a new analytical object called a rough path, that is not only the data of the path $(B_t)_{t>0}$ but also the iterated integrals

$$\mathbf{B} = \left(\int_{0\leq t_1\leq\ldots\leq t_n\leq t} \mathrm{d}B_{t_1}\otimes\ldots\otimes\mathrm{d}B_{t_n}\right)_{n\geq 1, t>0}$$

of the path against itself. A rough path lies in a precise algebraic structure, the space of iterated integrals described by Chen, and need to statify analytical bounds. Solutions of SDEs are then interpreted as the first component of the solution $\mathbf{X}$ of the associated rough differential equations (RDEs)

$$\mathrm{d}\mathbf{X}_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}\mathbf{B}_t$$

with different advantages. In particular, the solution $\mathbf{X}$ is a continuous function of the input rough path $\mathbf{B}$ whereas the solution of an SDE depends only measurably of the input path B. While rough paths allow to define the integral of 1-forms, this was extended in 2004 by Gubinelli with the notion of controlled paths where one looks for solutions whose variations are controlled by the Brownian rough path $\mathbf{B}$, and later with the notion of branched rough path with the Hopf algebra of trees as a generalisation of Chen's condition. This is the philosophy that led to the resolution of singular SPDEs with the notions of regularity structures as a generalisation of rough paths and paracontrolled calculus as a generalisation of the controlled paths approach.

Toolbox to investigate singular SPDEs

As far as SSPDEs are concerned, two different approaches emerged in 2013 following the rough paths philosophy : the regularity structures and the paracontrolled calculus. Again, the idea is that classic function spaces are too large for the problem to make sense hence the need for random subspaces built from the rough stochastic terms in which one can almost surely make sense of the equation.

Regularity structures were developped by Hairer in [39]. These are new structures to describe functions and distributions with a generalisation of the notion of local regularity. A function is of class C^k if it can be approximated by polynomials of degree k up to a quantified error. In a regularity structure, a certain set of reference distributions is given and the regularity of a distribution is measured by how well it can be approximated by these given objects. In general, one keeps the polynomials in the reference set in order to have a generalisation of the classical notion of regularity and add random distributions built from the noise. Solutions of PDEs are then described locally by generalised Taylor expansion with building block the reference distribution, this is the notion of modelled distributions. The local expansions should agree when based on different points, this gives algebraic condition that are encoded by Hopf algebra. If they also satisfy appropriate analytical conditions, one can reconstruct a global object based on the local expansions. To solve a singular SPDEs, one first constructs a random regularity structure from the stochastic rough term and then interpret the equation almost surely within this framework. The whole theory as a black box to solve SSPDEs was developped in the four following works.

- In his first paper [39], Hairer introduced the notion of regularity structures and solved the first examples of singular SPDEs.

- In [18], Bruned, Hairer and Zambotti set up the algebraic framework of renormalisation within regularity structures.

- Chandra and Hairer proved in [23] the convergence of the renormalisation procedure in the framework of the precedent work.

- In [17], Bruned, Chandra, Chevyrev and Hairer explicited how the renormalisation procedure acts at the level of the equations.

The paracontrolled calculus was introduced by Gubinelli, Imkeller and Perkowski in [35]. Whereas regularity structures relies on a local description with Taylor-like expansions, paracontrolled calculus deals directly with global objects using harmonic analysis. In this framework, the regularity of a distribution is measured by the asymptotic behavior of its Fourier transform for large frequencies. Thus distributions are described by a frequency "scheme" also based on a familly of reference distributions obtained from the rough stochastic term. To investigate singular products, a precise decomposition of the product is provided with Bony's paraproduct based on the Paley-Littlewood decomposition. The singularity of a product fg is encoded in a resonant term $\mathsf{\Pi}(f,g)$ while the roughest part is given by the paraproducts $\mathsf{P}_f g$ and $\mathsf{P}_g f$ which are always well-defined. Different correctors and commutators are then used to make sense of the equation almost surely and this yields a family of stochastic singular processes one has to define, this is the renormalisation step. While it was originaly a first order calculus on the torus, it was extended to a higher order calculus on manifolds.

- In [35], Gubinelli, Imkeller and Perkowski developped the paracontrolled calculus for singular SPDEs using Fourier theory.

- In [6], Bailleul and Bernicot devised a new approach based on the heat semigroup to work on unbounded Riemannian manifolds.

- In [8], Bailleul, Bernicot and Frey sharpened their previous construction and provided a pair of intertwinned space-time paraproducts.

- In [7], Bailleul and Bernicot extended paracontrolled calculus to a higher order calculus using space-time paraproducts.

As the title suggests, this is the framework in which this thesis fits.

An interesting question is the extension of the study of SSPDEs to the manifold framework, for example to construct QFT in curved spacetimes. While regularity structures were used in [26] by Dahlqvist, Diehl and Driver to solve the PAM equation on a two-dimensional closed Riemannian manifold, it remains to see if it is possible to adapt it to higher dimensions. The main results for singular stochastic PDEs on manifolds are obtained within the framework of paracontrolled calculus. The theory of high order paracontrolled calculus is now able to deal with the analytical formulation of singular SPDEs such as the generalised (PAM) equation

$$\partial_t u - \Delta u = f(u)\xi$$

in dimension 3 with ξ a space white noise or the generalised (KPZ) equation

$$\partial_t u - \partial_x^2 u = f(u)\zeta + g(u)(\partial_x u)^2$$

in dimension 1 with ζ a space-time white noise and where Δ is the Laplace-Beltrami operator associated on a Riemannian manifold; this is the content of Chapter 2. One can also deal with the analytical formulation of the quasi-linear version of these PDEs with the method described in Chapter 3. This theory is also well suited to deal with Sobolev spaces and the study of random singular operators. The Chapters 4 and 5 respectively study the Anderson Hamiltonian and the random magnetic Laplacian with white noise magnetic field. The last two Chapters relies on the Anderson Hamiltonian to investigate associated problems. In Chapter 6, Strichartz inequalities are provided for the Schrödinger group and the wave semigroup associated to the Anderson Hamiltonian which allow a low-regularity solutions theory for the associated PDEs. Chapter 7 presents the polymer measure with white noise potential on a two-dimensional manifold and the infinitesimal of the Brox diffusion, a continuous analogue of Sinai's random walk in random environment. A detailled description of each Chapters is given at the end of this introduction.

The algebraic structure in which a rough paths lies is clear and given by the Hopf algebra of words, later extended with branched rough path and the larger Hopf algebra of rooted trees. However the algebraic mechanism behind singular SPDEs depends on the equation where a large number of terms appears, more and more as the equation gets more singular. In regularity structures, this is also dealt with using Hopf algebras of decorated rooted trees tailor-made for each equation. In paracontrolled calculus, such a mechanism is still to be understood and is current investigation.

High order paracontrolled calculus

The tools of paracontrolled calculus can be briefly described as follows. Given any distribution $f \in \mathcal{D}'(\mathbb{T}^d)$, one can consider its Paley-Littlewood decomposition

$$f = \sum_{n\geq 0} \Delta_n f$$

where each $\Delta_n f$ is smooth and localised in frequencies in an annulus of radius 2^n. This allows for example to define function spaces for negative exponent that generalise the classical notion of Hölder and Sobolev regularity with a measure of growth at infinity of $\Delta_n f$. It can also be used to describe products as

$$f \cdot g = \sum_{n,m\geq 0} \Delta_n f \cdot \Delta_m g = P_f g + \Pi(f,g) + P_g f$$

where the sum in $P_f g$ is restricted to $n < m - 1$ and $\Pi(f,g)$ is restricted to $|n-m| \leq 1$. The point is that the paraproduct $P_f g$ is always well-defined while the resonant term $\Pi(f,g)$ encodes the potential singularity. Using this approach, Gubinelli, Imkeller and Perkowski translated the idea of controlled paths into paracontrolled calculus by looking for solution of the form

$$u = P_{u'} X + u^\sharp$$

where X is a noise-dependent function and $(u', u^\sharp)$ as the new unknown; this is the paracontrolled interpretation of "u locally looks like X". In order to adapt this approach to the manifold setting, Bailleul and Bernicot considered the decomposition

$$f = \lim_{t\to 0} e^{-t\Delta} f = \int_0^1 -(t\Delta)e^{-t\Delta}\frac{\mathrm{d}t}{t} + e^{-\Delta} f.$$

Using Gaussian upper bounds for the heat kernel and its derivatives, this defines a continuous analogue of the Paley-Littlewood decomposition where $t^{-\frac{1}{2}} \simeq 2^n$ and yields Hölder and Sobolev spaces for scalar fields on manifolds. A refinement of this decomposition allows to define the paraproduct and the resonant term such that

$$f \cdot g = \mathsf{P}_f g + \mathsf{\Pi}(f,g) + \mathsf{P}_g f$$

where P and $\mathsf{\Pi}$ verifie the same important properties as their Fourier analogue P and Π. In fact, this is not only true for the Laplace-Beltrami operator but for any nice enough elliptic operator $L = -\sum_i V_i^2$. This gives a paracontrolled calculus tailor made for the study of PDEs involving this family of first order differential operators $(V_i)_i$. Indeed, one can introduce a new paraproduct $\widetilde{\mathsf{P}}$ intertwined with P via the relation

$$D \circ \widetilde{\mathsf{P}} = \mathsf{P} \circ D$$

for any differential operator D obtained from the V_i's. It enjoys the same properties as P and appear naturally in the weak formulation of PDEs involving D. There is a wide range of choice in the construction of P, $\widetilde{\mathsf{P}}$ and $\mathsf{\Pi}$ which make the paracontrolled calculus a flexible theory.

We now sketch the general method to solve singular PDEs using high order paracontrolled calculus. Consider an equation of the form

$$Du = f(u,\zeta)$$

with D a differential operator, ζ a rough stochastic term and $f(u,\zeta)$ an expression involving a singular product. Using paracontrolled calculus, one decomposes the right hand side assuming that u is of a particular form paracontrolled by a set of reference functions $\mathcal{T}$ depending on ζ, this is the notion of paracontrolled system $\widehat{u} = (u_a)_{a\in\mathscr{A}}$. Using correctors and commutators, the aim is to get an expression of the form

$$f(u,\zeta) = \sum_{\sigma\in\mathcal{T}_f} \mathsf{P}_{v_\sigma}\sigma$$

where $\mathcal{T}_f$ is a new set of noise-dependent functions dictated by the equation and v_σ are functionnals of the generalised unknown $\widehat{u}$. Using the paraproduct $\widetilde{\mathsf{P}}$ intertwined by D to describe the mild formulation, the equation rewrites

$$\sum_{\tau\in\mathcal{T}} \widetilde{\mathsf{P}}_{u_\tau}\tau = \sum_{\sigma\in\mathcal{T}_f} \widetilde{\mathsf{P}}_{v_\sigma}(D^{-1}\sigma).$$

Semilinear parabolic PDEs as (gPAM) or (gKPZ) corresponds to $D = \partial_t - \Delta$ and this defines recursively the set $\mathcal{T}$ by identifying terms according to their Hölder regularity. Then one can perform a fixed point to solve the equation almost surely once the set $\mathcal{T}$ is actually constructed through a renormalisation procedure. This is the method set in place in Chapter 2 with the following steps.

- Start from a paracontrolled system $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ for a given unkown set $\mathcal{T}$ of noise-dependent reference functions.
- Use the paracontrolled calculus toolbox to construct recursivly $\mathcal{T}$ and formulate the equation in a suitable space of functions paracontrolled by $\mathcal{T}$.
- Solve the equation via a fixed point Theorem.

One of the simplest singular SPDE is the heat equation with multiplicative noise

$$\partial_t u - \Delta u = u\zeta$$

with ζ a stochastic source term that belongs almost surely to $\mathcal{C}^{\alpha-2}$ with $\alpha > 0$. Schauder estimates give the hint that the solution should be α-Hölder thus consider $u \in \mathcal{C}^\alpha$ described by a paracontrolled system $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ with reference set $\mathcal{T}$, see Chapter 2 for the details. Roughly, u is described as

$$u = \sum_{\tau\in\mathcal{T}} \widetilde{\mathsf{P}}_{u_\tau}\tau + u^\sharp$$

and all "paraderivatives" are also paracontrolled as

$$u_{\tau_1} = \sum_{\tau_2\in\mathcal{T}} \widetilde{\mathsf{P}}_{u_{\tau_1\tau_2}}\tau_2 + u^\sharp_{\tau_1}$$

up to an order depending on the regularity of $\tau_1 \in \mathcal{T}$. Using the paracontrolled toolkit, we get the decomposition of the product

$$u\zeta = \mathsf{P}_u\zeta + \mathsf{P}_\zeta u + \mathsf{\Pi}(u,\zeta) = \sum_{\sigma\in\mathcal{T}'} \mathsf{P}_{\phi_\sigma(\widehat{u})}\sigma$$

with an explicit set $\mathcal{T}'$ depending on ζ and $\mathcal{T}$ and ϕ_σ explicit functionnals of the unknown $\widehat{u}$. The equation then rewrites

$$\sum_{\tau\in\mathcal{T}} \widetilde{\mathsf{P}}_{u_\tau}\tau = \sum_{\sigma\in\mathcal{T}'} \widetilde{\mathsf{P}}_{\phi_\sigma(\widehat{u})}(\mathscr{L}^{-1}\sigma)$$

and this defines a unique set $\mathcal{T}$ of reference functions. Once each functions in $\mathcal{T}$ is constructed through a renormalisation procedure, this shows that the space of paracontrolled system is stable by the fixed point formulation. Using an adapted norm for each remainders $(u^\sharp_a)_{a\in\mathscr{A}}$, one gets a contraction for a small horizon time. The rougher the noise is, the higher the order of the system needs to be. Remark that the algebraic mechanism that lead to the construction of $\mathcal{T}$ for general sub-critcal equation is still under investigation in the framework of paracontrolled calculus.

This approach can also be used to study singular random operator. An interesting example is the Anderson Hamiltonian

$$H = \Delta + \xi$$

with ξ a space white noise. It appears for example in the formulation of associated heat, Schrödinger or wave equations. One is interested in its spectral properties hence in the solutions (λ, u) to

$$\Delta u + u\xi = \lambda u.$$

There are a lot of interesting questions like the regularity of solutions, localisation of their support, asymptotics and more. In dimension $d \geq 2$, the product $u\xi$ is singular hence one needs to find a proper interpration for the operator. Following the singular SPDEs philosophy, the idea is to construct a random domain through a renormalisation procedure to get an unbounded operator in L^2. This gives a self-adjoint operator with discrete spectrum and we can study more precisely its properties, this is the content of Chapter 4. In particular, we provide an almost sure Weyl-type law. We also get Strichartz inequalities for the associated Schrödinger and wave semigroups on two-dimensional manifolds using that similar result holds for the Laplace-Beltrami operator with only an arbitrary small loss of regularity, this is the content of Chapter 6.

The Anderson Hamiltonian can be interpreted as an eletric Laplacian with white noise as eletric field. Following the introduction and study of the magnetic Laplacian in the 1970's independantly by Simon and Helffer, one can consider the magnetic Laplacian with white noise as magnetic field. The magnetic Laplacien on the two-dimensional torus is given by

$$(i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2$$

with $A = (A_1, A_2)$ the potential vector field with the induced magnetic field being

$$B = \nabla \times A = \partial_2 A_1 - \partial_1 A_2.$$

As for the Anderson operator, paracontrolled calculus allows the study of the random magnetic Laplacian with space white noise as magnetic field. The operator is also self-adjoint with pure point spectrum with an almost sure Weyl-type law. The method illustrates the flexibility of the approach which allows to deal with a general class of operators of the form

$$\Delta + a_1 \cdot \nabla + a_2$$

with a_1, a_2 random fields. The case $a_2 = 0$ corresponds for example to the infinitesimal generator of diffusion given by the SDE

$$\mathrm{d}X_t = a_1(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t$$

with B a Brownian motion.

Outline of the thesis

We construct the high order paracontrolled calculus on manifolds based on the heat semigroup. It allows to solve different elliptic, parabolic and hyperbolic PDEs on manifolds involving renormalisation procedures of singular stochastic terms. Chapter 1 deals with the construction of the paracontrolled calculus using the heat semigroup. In Chapters 2 to 6, different examples of parabolic, elliptic and dispersive SSPDEs are studied. Finally, Chapter 7 presents two different random models using singular stochastic operators defined through the paracontrolled calculus.

Chapter 1 : Construction of the paracontrolled calculus.

The tools of high order paracontrolled calculus on a compact manifold based on the heat semigroup are constructed in the space setting with the two paraproducts P and $\widetilde{\mathsf{P}}$. The first one allows to decompose the product operation as

$$ab = \mathsf{P}_a b + \mathsf{P}_b a + \Pi(a,b)$$

where $\mathsf{P}_a b$ and $\mathsf{P}_b a$ are well defined for any distributions while the resonant term $\Pi(a,b)$ captures the possible singularity of the product. The second one is intertwined with P via a relation of the type

$$D \circ \widetilde{\mathsf{P}} = \mathsf{P} \circ D$$

where D is any differential operator of interest and allows to express weak formulation of PDEs involving a product described by P. The paraproduct P is built from the heat semigroup associated to a nice enough family of first order differential operators $(V_i)_{1\le i\le d}$ and D can be any operator built from this family. In the case of spacetime paracontrolled calculus, one can also consider the time derivative ∂_t. In our applications, we take $D = -\Delta$ or $D = \partial_t - \Delta$. In order to get around singular products, one needs continuity estimates on correctors of the type

$$\mathsf{C}(a_1,a_2,b) := \Pi\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - a_1 \Pi(a_2,b)$$

together with its refined and iterated versions. Other correctors and commutators are also needed to investigate well-defined terms that are not in an adapted form to solve the problem under consideration. For example, one might need the commutator

$$\mathsf{D}(a_1,a_2,b) := \Pi\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - \mathsf{P}_{a_1} \Pi(a_2,b)$$

or the merging operator

$$\mathsf{R}(a,b,c) := \mathsf{P}_a \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} c.$$

Twisted version of these operators can be considered for example to deal with

$$(\partial u)^2 = 2\mathsf{P}_{\partial u} \partial u + \Pi(\partial u, \partial u).$$

One of the major advantages of paracontrolled calculus based on the heat semigroup is that the notion of paraproducts and resonant terms is very flexible. In particular, operators such as

$$(a,b) \mapsto \mathsf{P}_{Da} D' b, \Pi(Da, D'b)$$

are easily dealt with for differential operators D, D' built from the same V_i's one uses to construct the paracontrolled tools.

Chapter 2 : Semilinear parabolic singular stochastic PDEs.

The high order paracontrolled calculus can be used to solve a large class of semilinear parabolic singular stochastic PDEs, including the generalised (PAM) equation in dimension 3

$$\partial_t u - \Delta u = f(u)\xi$$

with ξ a space white noise and the generalised (KPZ) equation in dimension $1+1$

$$\partial_t u - \partial_x^2 u = f(u)\zeta + g(u)(\partial_x u)^2$$

with ζ a spacetime white noise. The method relies on the notion of paracontrolled systems $(u_a)_{a\in\mathscr{A}}$ given a set of noise-dependent reference functions $\mathcal{T}$ satisfying a triangular system

$$u_a = \sum_{|a\tau|\le n\alpha} \widetilde{\mathsf{P}}_{u_{a\tau}}\tau + u_a^\sharp$$

where $|\cdot|$ corresponds to the Hölder regularity of the different functions. The set $\mathcal{T}$ should be constructed through a renormalisation procedure and the PDEs are solved almost surely via a fixed point on the random space of remainders $(u_a^\sharp)_{a\in\mathscr{A}}$. We only present the fixed point part of the method.

Chapter 3 : Quasilinear parabolic singular stochastic PDEs.

- I. Bailleul and A. Mouzard, *Paracontrolled calculus for quasilinear singular PDEs*, arXiv:1912.09073, (2019).

The high order paracontrolled calculus setting is developped further to deal with the analytic part of the study of quasilinear singular PDEs. Continuity results are proved for a number of operators for that purpose and we use infinite dimensional paracontrolled systems based on a finite algebraic strucutre. We consider any quasilinear PDE

$$\partial_t u - d(u)\Delta u = f(u,\xi)$$

that one can solve using high order paracontrolled calculus, including (gPAM) and (gKPZ). The equation is reformulated as

$$\partial_t u + Lu = f(u,\xi) + \varepsilon(u,\cdot)Lu + a_i(u,\cdot)V_i u$$

with L an elliptic second order differential operator in Hörmander form in order to use an adapted paracontrolled calculus depending on the initial condition. Introducting new correctors and commutators associated to the product involving L and the V_i's, we are able to formulate the equation as a fixed point on a space of paracontrolled systems. Due to the second order term on the right hand side, one can not find a finite set of reference functions $\mathcal{T}$ and has to consider integer "decoration" on the finite set given by the semilinear equation. Going from finite to infinite dimensionnal algebraic space also appears in the other approaches developped for these equations; this seems to be the quasilinear effect.

Chapter 4 : The Anderson Hamiltonian.

- A. Mouzard, *Weyl law for the Anderson Hamiltonian on a two-dimensional manifold*, arXiv:2009.03549, (2020).

This Chapter deals with the construction of the Anderson Hamiltonian

$$H = -\Delta + \xi$$

on a two-dimensional manifold with ξ a space white noise using high order paracontrolled calculus. Due to the singularity of the operator, it involves a renormalisation procedure and it yields a self-adjoint operator with pure point spectrum $\big(\lambda(\Xi)\big)_{n\geq 1}$. The Hölder regularity of the eigenfunctions is computed and we provide lower and upper bounds on its eigenvalues of the form

$$\lambda_n - m_\delta^2(\Xi) \leq \lambda_n(\Xi) \leq (1+\delta)\lambda_n + m_\delta^1(\Xi)$$

for any $\delta \in (0,1)$, explicit constant m_δ^1, m_δ^2 depending on the enhanced noise Ξ and $(\lambda_n)_{n\geq 1}$ the eigenvalues of the Laplacian. In particular, it implies an almost sure Weyl-type law for H for the form

$$\lim_{\lambda\to\infty} \lambda^{-1}\big|\{\lambda_n(\Xi) \leq \lambda\}\big| = \frac{\mathrm{Vol(M)}}{4\pi}.$$

Chapter 5 : The random magnetic Laplacian.

- L. Morin and A. Mouzard, *2D random magnetic Laplacian with white noise magnetic field*, arXiv:2101.05020, (2021).

We construct the random magnetic Laplacian

$$H = (i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2$$

on the two-dimensional torus with $A = (A_1, A_2) \in \mathcal{C}^{\alpha-1} \times \mathcal{C}^{\alpha-1}$ a rough random magnetic potential where $\alpha < 1$ using the paracontrolled calculus. The potential is taken such that it yields as magnetic field the space white noise with

$$A = \nabla^\perp \varphi \quad \text{where} \quad \varphi = \Delta^{-1}\xi.$$

After a renormalisation procedure, it is a self-adjoint operator with pure point spectrum. We also provide lower and upper bounds on its eigenvalues with an almost sure Weyl-type law. In particular, this is an example of operator of the form

$$-\Delta + a_1 \cdot \nabla + a_2$$

with random scalar fields $a_1, a_2 : \mathbb{T}^2 \to \mathbb{R}$ rougher than the classical theory can deal with.

Chapter 6 : Dispersives singular SPDEs.

- A. Mouzard and I. Zachhuber, *Strichartz inequalities with white noise potential on compact surfaces*, arXiv:2104.07940, (2021).

We solve nonlinear dispersive PDEs with a multiplicative white noise using the construction of the Anderson Hamiltonian which allows for example the interpretation of the Schrödinger equation

$$i\partial_t u + \Delta u = u\xi + |u|^2 u$$

as

$$i\partial_t u = Hu + |u|^2 u.$$

The different properties of H allows to get strong and energy solutions to this equation as well as to the wave equation on a compact surface without boundary or with Dirichlet boundary conditions. In the deterministic case, the Strichartz inequalities of the form

$$\|e^{it\Delta}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha}$$

allow to get a low-regularity solution theory for these equations. We prove such results for the Anderson Hamiltonian, that is estimates of the form

$$\|e^{itH}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha}$$

which allows to get local well-posedness in low regularity Sobolev spaces for the Schrödinger and wave equations with cubic nonlinearity and multiplicative noise on compact surfaces with or without boundary. In particular, this yields bounds on the L^q-norm of the eigenvalues of the Anderson Hamiltonian and its spectral projectors.

Chapter 7 : Diffusions in disordered media.

- A. Mouzard, *The continuum polymer measure with white noise potential on compact surfaces*, in preparation.
- A. Mouzard, *The Brox diffusion on a circle and its generator*, in preparation.

We present two models of diffusions in disordered media using singular stochastic operators. The first one is the polymer measure with white noise potential ξ formally described by

$$\mathbf{V}(\mathrm{d}X) = \frac{1}{Z_T} e^{-\frac{1}{2}\int_0^T \xi(X_s)\mathrm{d}s}\mathbf{W}(\mathrm{d}X)$$

where $\mathbf{W}$ is the Wiener measure on $C([0,T],M)$. Since the white noise is only a distribution, the term $\xi(X_s)$ does not make sense and this Gibbsian formalism can not be used. This can be understood in view of the fact that the polymer measure will actually be singular with the Wiener measure hence can not have a density with respect to it. Our construction will be based on the so-called intrinsic Feynman-Kac semigroup associated to the Anderson Hamiltonian and is related to the diffusion

$$\mathrm{d}X_t = \nabla(\log\Psi)\mathrm{d}t + \mathrm{d}B_t$$

with Ψ the Anderson gound state. The second model is the Brox diffusion formally given by SDE

$$\mathrm{d}X_t = \xi(X_t)\mathrm{d}t + \mathrm{d}B_t$$

where the drift is singular and given by a space white noise ξ in one dimension. Its infinitesimal generator is formally given by

$$-\frac{1}{2}\Delta + \xi\cdot\nabla$$

which can be constructed using the heat semigroup paracontrolled calculus. In particular, this gives a new approach to the study of the Brox diffusion as well as its construction on the circle where one can not use the self-similarity property. Since the study of these two models is based on ongoing works, we only present the general ideas.

Chapter 1

Paracontrolled calculus

The term “paraproduct” was firs used by Coifman and Meyer in [25] untitled “Au delà des opérateurs pseudo-différentiels”. This comes from the Greek word “para” which translates to “au delà” or “beyond”. This was also used by Bony in [14] in order to adapt pseudo-differentials operators to deal with nonlinear PDEs. Since then, a paraproduct has been used in different contexts for a bilinear object going beyond the usual product depending on the goal. In our framework, the paraproduct is a tool to decompose a product between two distributions in order to understand where the singularity comes from. While a product between an α-Hölder distribution f and a β-Hölder one f is well defined if $\alpha+\beta>0$, the paraproducts $\mathsf{P}_f g$ and $\mathsf{P}_g f$ always make sense. Therefore, the potential singularity of a product is encoded in the remainder

$$\Pi(f,g) := fg - \mathsf{P}_f g - \mathsf{P}_g f.$$

Moreover, the paraproduct has the particularity that $\mathsf{P}_f g$ behaves locally like g if f is actually a function. This is the important property that led Imkeller, Perkowski and Gubinelli to the introduction of paracontrolled calculus as an equivalent of Gubinelli's controlled path in an infinite dimensional setting. This allows the understanding of the infinite quantity of the singular product in play for a class of stochastic partial differential equations and thus to go beyond it using this paraproduct :

“ *To infinity and beyond* ”.

On the torus, Fourier analysis yields an approximation of any distributions in $\mathcal{D}(\mathbb{T}^d)$. One can then measure its regularity using its Paley-Littlewood decomposition which can also be used to construct Bony's paraproduct. On a manifold M, the heat semigroup $P := (e^{tL})_{t>0}$ associated to a nice enough second order differential operator L can be used to regularise distributions in $\mathcal{D}'(M)$. One can then consider the Calderón décomposition as an analogue of the Paley-Littlewood decomposition with a continuous scaling parameter and $Q_t := -t\partial_t P_t$ acting like a localizer on “frequency” of order $t^{-\frac{1}{2}}$. After giving the geometric framework, we introduce the standard families of operators we shall use to define the Besov spaces on M. We then construct the paraproducts P and $\widetilde{\mathsf{P}}$ with the tools of high order paracontrolled calculus to study elliptic PDEs. Finally, we discuss different generalisations including the space-time framework to study parabolic PDEs, the unbouded spatial setting and a time weight to deal with rougher initial conditions.

In this Chapter, we only present the construction of the paracontrolled calculus based on the heat semigroup for space distributions in order to simplify the technical details. We give a brief outline of the spacetime calculus in Section 1.6 and refer for the details to the works [8, 7] by Bailleul, Bernicot and Frey where it was first introduced. The construction in the spatial setting is from the work [50].

1.1 – Geometric framework

Let (M,d,μ) be a complete volume doubling measured Riemannian manifold. We assume M compact to avoid the use of spatial weights; everything in this section should work in the unbounded setting of [8]. All the kernels we consider are with respect to this measure μ. Let $(V_i)_{1\le i\le d}$ be a family of smooth vector fields identified with first order differential operators on M. Consider the associated second order operator L given by

$$L = -\sum_{i=1}^{d} V_i^2.$$

We assume that L is *elliptic*. In particular, it implies that the vector fields $(V_i)_{1\le i\le d}$ span smoothly at every point of M the tangent space and the existence of smooth functions $(\gamma_i)_{1\le i\le d}$ such that for any $f\in C^1(M,\mathbb{R})$ and $x\in M$, we have

$$\nabla f(x) = \sum_{i=1}^{d} \gamma_i(x) V_i(f)(x) V_i(x).$$

It also implies that L is sectorial in L^2 with kernel the constant functions, it has a bounded H^∞-calculus on L^2 and $-L$ generates a holomorphic semigroup $(e^{-tL})_{t>0}$ on L^2, see [30]. Given any collection $I=(i_1,\dots,i_n)\in\{1,\dots,d\}^n$, we denote by $V_I := V_{i_n}\dots V_{i_1}$ the differential operator of order $|I| := n$. Under the smoothness and ellipticity conditions, the semigroup has regularity estimate at any order, that is $(t^{\frac{|I|}{2}}V_I)e^{-tL}$ and $e^{-tL}(t^{\frac{|I|}{2}}V_I)$ have kernels $K_t(x,y)$ for any $t>0$ and $x,y\in M$ that satifies the Gaussian estimates

$$\big|K_t(x,y)\big| \lesssim \mu\big(B(x,\sqrt{t})\big)^{-1} e^{-c\frac{d(x,y)^2}{t}}$$

and for $x'\in M$

$$\big|K_t(x,y) - K_t(x',y)\big| \lesssim \frac{d(x,x')}{\sqrt{t}} \mu\big(B(x,\sqrt{t})\big)^{-1} e^{-c\frac{d(x,y)^2}{t}}$$

for $d(x,x')\le\sqrt{t}$ and a constant $c>0$. The range of application contains the case of a bounded domain with its Laplacian associated with periodic or Dirichlet boundary conditions if the boundary is sufficiently regular, see again [30]. In particular, the Laplacien can indeed be written in the Hörmander form, see Strook's book [57] for example.

The operator $L : \mathcal{H}^2 \subset L^2 \to L^2$ is not invertible since its kernel contains constant function however it is invertible up to a smooth error term. Indeed, setting

$$L^{-1} := \int_0^1 e^{-tL}\mathrm{d}t,$$

we have $L\circ L^{-1} = \mathrm{Id}$ up to the regularising operator e^{-L}. In the litterature, L^{-1} is often refered as a parametrix.

1.2 – Approximation theory

All computations below make sense for a choice of large enough integers b and ℓ that are fixed in any application, we also assume b even. Given $x, y \in M$ and $t \in (0,1]$, we define the kernel

$$\mathcal{G}_t(x,y) := \frac{1}{\mu\left(B(x,\sqrt{t})\right)} \left(1 + c\,\frac{d(x,y)^2}{t}\right)^{-\ell}$$

with $c > 0$ a constant. We do not emphasize the dependance on the positive constant c and abuse notation by writing the same letter $\mathcal{G}_t$ for two functions corresponding to two different values of the constant. We have for any $s, t \in (0,1]$

$$\int_M \mathcal{G}_t(x,y)\mathcal{G}_s(y,z)\mathrm{d}y \lesssim \mathcal{G}_{t+s}(x,z).$$

A choice of constant ℓ large enough ensures that

$$\sup_{t\in(0,1]} \sup_{x\in M} \int_M \mathcal{G}_t(x,y)\mathrm{d}y < \infty.$$

This implies that any linear operator with a kernel pointwisely bounded by $\mathcal{G}_t$ is bounded in $L^p(M)$ for every $p \in [1,\infty]$. The family $(\mathcal{G}_t)_{t\in(0,1]}$ is our reference kernel for Gaussian operator due to the singularity as t goes to 0; this is the letter 'G' in the following definition.

Definition. *We define G as the set of families $(P_t)_{t\in(0,1]}$ of linear operator on M with kernels pointwisely bounded by*

$$|K_{P_t}(x,y)| \lesssim \mathcal{G}_t(x,y)$$

given any $x, y \in M$.

We consider two such families of operators $(Q_t^{(b)})_{t\in(0,1]}$ and $(P_t^{(b)})_{t\in(0,1]}$ defined as

$$Q_t^{(b)} := \frac{(tL)^b e^{-tL}}{(b-1)!} \quad \text{and} \quad -t\partial_t P_t^{(b)} = Q_t^{(b)}$$

with $P_0^{(b)} = \mathrm{Id}$. In particular, there exist a polynomial p_b of degree $(b-1)$ such that $P_t^{(b)} = p_b\,(tL)\,e^{-tL}$ and $p_b(0) = 1$. The family $(P_t)_{t\in(0,1]}$ regularises distributions while the family $(Q_t)_{t\in(0,1]}$ is a kind of localizer on 'frequency' of order $t^{-\frac{1}{2}}$ as one can see with the parabolic scaling of the Gaussian kernel. In the flat framework of the torus, this can be explicitly written using Fourier theory. These tools also enjoy cancellation properties as Fourier projectors however it is not as precise since the operators involved here are not locally supported. For example, the following simple computation shows that the composition

$$Q_t^{(b)} \circ Q_s^{(b)} \simeq \left(\frac{ts}{(t+s)^2}\right)^b Q_{t+s}^{(2b)}$$

is small for $s \ll t$ or $t \ll s$ but not equal to 0. The importance of the parameter b appears here as a 'degree' of cancellation. One can also see that in the fact that for any polynomial function p of degree less than $2b$ in the flat case, we have $P_t^{(b)}p = p$ and $Q_t^{(b)}p = 0$ for any $t \in (0,1]$. We now define the standard family of Gaussian operators with cancellation that we shall use in this work.

Definition. *Let $a \in [\![0, 2b]\!]$. We define the standard collection of operators with cancellation of order a as the set StGC^a of families*

$$\Big((t^{\frac{|I|}{2}} V_I)(tL)^{\frac{j}{2}} P_t^{(c)} \Big)_{t \in (0,1]}$$

with I, j such that $a = |I| + j$ and $c \in [\![1, b]\!]$. These operators are uniformly bounded in $L^p(M)$ for every $p \in [1, \infty]$ as functions of the parameter $t \in (0, 1]$. In particular, a standard family of operator $Q \in \mathsf{StGC}^a$ can be seen as a bounded map $t \mapsto Q_t$ from $(0, 1]$ to the space of bounded linear operator on $L^p(M)$. We also set

$$\mathsf{StGC}^{[0,2b]} := \bigcup_{0 \le a \le 2b} \mathsf{StGC}^a.$$

Since the first order differential operators V_i do not a priori commute with each other, they do not commute with L and we introduce the notation

$$\big(V_I \phi(L)\big)^{\bullet} := \phi(L) V_I$$

for any function ϕ such that $\phi(L)$ is defined in order to state the following cancellation property. This is not related to any notion of duality in general. In particular, L is not supposed self-adjoint here.

Proposition 1.1. *Given $a, a' \in [\![0, 2b]\!]$, let $Q^1 \in \mathsf{StGC}^a$ and $Q^2 \in \mathsf{StGC}^{a'}$. Then for any $s, t \in (0, 1]$, the composition $Q_s^1 \circ Q_t^{2\bullet}$ has a kernel pointwisely bounded by*

$$\begin{aligned}
\big|K_{Q_s^1 \circ Q_t^{2\bullet}}(x, y)\big| &\lesssim \left(\Big(\frac{s}{t}\Big)^{\frac{a}{2}} \mathbb{1}_{s<t} + \Big(\frac{t}{s}\Big)^{\frac{a'}{2}} \mathbb{1}_{s \ge t} \right) \mathcal{G}_{t+s}(x, y) \\
&\lesssim \left(\frac{ts}{(t+s)^2} \right)^{\frac{a}{2}} \mathcal{G}_{t+s}(x, y)
\end{aligned}$$

with $a = \min(a, a')$.

Proof: Let $t \in (0, 1]$. We have

$$Q_t^1 = t^{\frac{a}{2}} V_I L^{\frac{j}{2}} P_t^{(c)} \quad \text{and} \quad Q_t^2 = t^{\frac{a'}{2}} V_{I'} L^{\frac{j'}{2}} P_t^{(c')}$$

with $c, c' \in [\![1, b]\!]$, $a = |I| + j$ and $a' = |I'| + j'$. For any $t, s \in (0, 1]$, the composition is given by

$$\begin{aligned}
Q_s^1 \circ Q_t^{2\bullet} &= s^{\frac{a}{2}} t^{\frac{a'}{2}} V_I L^{\frac{j+j'}{2}} P_s^{(c)} P_t^{(c')} V_{I'} \\
&= \frac{s^{\frac{a}{2}} t^{\frac{a'}{2}}}{(t+s)^{\frac{a+a'}{2}}} (t+s)^{\frac{a+a'}{2}} V_I L^{\frac{j+j'}{2}} P_s^{(c)} P_t^{(c')} V_{I'}
\end{aligned}$$

and this yields

$$\begin{aligned}
K_{Q_s^1 \circ Q_t^{2\bullet}}(x, y) &\lesssim \frac{s^{\frac{a}{2}} t^{\frac{a'}{2}}}{(t+s)^{\frac{a+a'}{2}}} \ \mathcal{G}_{t+s}(x, y) \\
&\lesssim \Big\{ \Big(\frac{s}{t}\Big)^{\frac{a}{2}} \mathbb{1}_{s<t} + \Big(\frac{t}{s}\Big)^{\frac{a'}{2}} \mathbb{1}_{s \ge t} \Big\} \mathcal{G}_{t+s}(x, y).
\end{aligned}$$

The last estimate follows from a direct computation.

$\square$

Operators with cancellation but not in this standard form also appear in the description of solutions to PDEs. This is the role of the set GC^a of the following definition.

Definition. *Let $a \in [\![0, 2b]\!]$. We define the subset $\mathsf{GC}^a \subset \mathsf{G}$ as families $(Q_t)_{t\in(0,1]}$ of operators with the following cancellation property. For any $s, t \in (0, 1]$ and standard family $S \in \mathsf{StGC}^{a'}$ with $a' \in [\![a, 2b]\!]$, the operator $Q_s \circ S_t^\bullet$ has a kernel pointwisely bounded by*

$$\left|K_{Q_s \circ S_t^\bullet}(x, y)\right| \lesssim \left(\frac{ts}{(t+s)^2}\right)^{\frac{a}{2}} \mathcal{G}_{t+s}(x, y).$$

The set StGC can be used to define Besov spaces on a manifold. For any $f \in L^p(M)$ with $p \in [1, \infty[$ or $f \in C(M)$, we have the following reproducing Calderón formula

$$f = \lim_{t\to 0} P_t^{(b)} f = \int_0^1 Q_t^{(b)} f \frac{\mathrm{d}t}{t} + P_1^{(b)} f.$$

We interpret it as an analogue to the Paley-Littlewood decomposition of f on a manifold but with a continuous parameter. Indeed, the measure $\frac{\mathrm{d}t}{t}$ gives unit mass to the dyadic intervals $[2^{-(i+1)}, 2^{-i}]$ with the operator $Q_t^{(b)}$ as a kind of multiplier roughly localized at frequencies of size $t^{-\frac{1}{2}}$. This motivates the following definition.

Definition. *Given any $p, q \in [1, \infty]$ and $\alpha \in (-2b, 2b)$, we define the Besov space $\mathcal{B}^\alpha_{p,q}(M)$ as the set of distribution $f \in \mathcal{D}'(M)$ such that*

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} := \left\|e^{-L} f\right\|_{L^p(M)} + \sup_{\substack{Q\in\mathsf{StGC}^k \\ |\alpha|<k\leq 2b}} \left\|t^{-\frac{\alpha}{2}} \|Q_t f\|_{L^p_x}\right\|_{L^q(t^{-1}\mathrm{d}t)} < \infty.$$

Remark : *As far as regularity is concerned, a limitation appears with this definition of $\mathcal{B}^\alpha_{p,q}$ since we can only work with regularity exponent $\alpha \in (-2b, 2b)$. This restriction is only technical and b can be taken as large as needed. It can be seen for example in the flat case with the fact that $Q_t^{(b)} p = 0$ for any $t > 0$ and polynomial function p of order less than $2b$.*

The Hölder spaces $\mathcal{C}^\alpha := \mathcal{B}^\alpha_{\infty,\infty}$ and Sobolev spaces $\mathcal{H}^\alpha := \mathcal{B}^\alpha_{2,2}$ are of particular interest with

$$\|f\|_{\mathcal{C}^\alpha} := \|e^{-L} f\|_{L^\infty} + \sup_{\substack{Q\in\mathsf{StGC}^k \\ |\alpha|<k\leq 2b}} \sup_{t\in(0,1]} t^{-\frac{\alpha}{2}} \|Q_t f\|_{L^\infty_x}$$

and

$$\|f\|_{\mathcal{H}^\alpha} := \|e^{-L} f\|_{L^2} + \sup_{\substack{Q\in\mathsf{StGC}^k \\ |\alpha|<k\leq 2b}} \left(\int_0^1 t^{-\alpha} \|Q_t f\|^2_{L^2_x} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}.$$

This is indeed a generalisation of the classical Hölder spaces as stated in the following Proposition. We shall denote C^α the classical spaces of Hölder functions with the norm

$$\|f\|_{C^\alpha} := \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

for $0 < \alpha < 1$. Note that for any integer regularity exponent, $\mathcal{C}^\alpha \neq C^\alpha$ since $\mathcal{C}^1$ contains the space of Lipschitz functions. The proof of the following Proposition is left to the reader, it works exactly as Proposition 5 in [8]. This mainly relies on the fact that Q_t is localised on spatial scale $|x-y| \simeq \sqrt{t}$ if it encodes cancellations.

Proposition. *For any $\alpha \in (0,1)$, we have $\mathcal{C}^\alpha = C^\alpha$ and the norms $\|\cdot\|_{\mathcal{C}^\alpha}$ and $\|\cdot\|_{C^\alpha}$ are equivalent.*

We have an analogue result for Sobolev spaces however one has to be careful in the case of a manifold with boundary. The semigroup is obtained with Dirichlet conditions hence the associated Sobolev spaces are the analogue of the classical H_0^α spaces. We keep the notation $\mathcal{H}^\alpha$ but the reader should keep that in mind.

Given a distribution $f \in \mathcal{C}^\alpha$ and $Q \in \mathsf{StGC}^k$, we have by definition a bound for $\|Q_t f\|_\infty$ only for $|\alpha| < k$. If f is a distribution and not a function, the quantity diverges and we still have the estimate for all k; this will be important to keep an accurate track of the regularity. The same holds for negative Sobolev spaces.

Proposition 1.2. *Let $-2b < \alpha < 0$ and $P \in \mathsf{StGC}^k$ with $k \in [\![0,b]\!]$. For $f \in \mathcal{C}^\alpha$, we have*

$$\sup_{t\in(0,1]} t^{-\frac{\alpha}{2}} \|P_t f\|_{L^\infty} \lesssim \frac{1}{k-\alpha} \|f\|_{\mathcal{C}^\alpha}.$$

For $f \in \mathcal{H}^\alpha$, we have

$$\|t^{-\frac{\alpha}{2}} \|P_t f\|_{L^2_x}\|_{L^2(t^{-1}\mathrm{d}t)} \lesssim \frac{1}{k-\alpha} \|f\|_{\mathcal{H}^\alpha}.$$

Proof : Since $P \in \mathsf{StGC}^k$ with $k \in [\![0,2b]\!]$, there exist $I=(i_1,\dots,i_n), j \in \mathbb{N}$ and $c \in [\![1,b]\!]$ such that $k = |I| + j$ and

$$P_t = (t^{\frac{|I|}{2}} V_I)(tL)^{\frac{j}{2}} P_t^{(c)}.$$

If $|\alpha| < k$, the result holds by definition of $\mathcal{C}^\alpha$. If $|\alpha| \geq k$, we have

$$\begin{aligned} P_t f &= (t^{\frac{|I|}{2}} V_I)(tL)^{\frac{j}{2}} \left(\int_t^1 Q_s^{(c)} f \frac{\mathrm{d}s}{s} + P_1^{(c)} f \right) \\ &= \int_t^1 \left(\frac{t}{s}\right)^{\frac{k}{2}} (s^{\frac{|I|}{2}} V_I)(sL)^{\frac{j+c}{2}} P_s^{(1)} f \frac{\mathrm{d}s}{s} + R_t f \\ &= \int_t^1 \left(\frac{t}{s}\right)^{\frac{k}{2}} Q_s f \frac{\mathrm{d}s}{s} + R_t f \end{aligned}$$

with $Q_s := (s^{\frac{|I|}{2}} V_I)(sL)^{\frac{j+c}{2}} P_s^{(1)} \in \mathsf{StGC}^{k+c}$ and $R_t := (t^{\frac{|I|}{2}} V_I)(tL)^{\frac{j}{2}} P_1^{(c)}$. The term $R_t f$ is bounded because of the smoothing operator $P_1^{(c)}$. Since $c \geq 1$, Q belongs at least to StGC^{k+1} hence if $|\alpha| < k+1$ we have

$$\begin{aligned} t^{-\frac{\alpha}{2}} \|P_t f\|_{L^\infty} &\leq t^{-\frac{\alpha}{2}} \int_t^1 \left(\frac{t}{s}\right)^{\frac{k}{2}} \|Q_s f\|_{L^\infty} \frac{\mathrm{d}s}{s} \\ &\leq \|f\|_{\mathcal{C}^\alpha} \int_t^1 \left(\frac{t}{s}\right)^{\frac{k-\alpha}{2}} \frac{\mathrm{d}s}{s} \\ &\leq \|f\|_{\mathcal{C}^\alpha} \frac{2}{k-\alpha} \end{aligned}$$

and this yields the result using that $\alpha < 0 \leq k$ hence $k - \alpha > 0$. If $|\alpha| \geq k+1$, using the same integral representation for Q and an induction completes the proof of the L^∞-estimate. For the L^2-estimate, we interpolate between L^1 and L^∞ as in Appendix A.1 to get

$$\begin{aligned} \|t^{-\frac{\alpha}{2}}\|P_t f\|_{L^2}\|_{L^2(t^{-1}\mathrm{d}t)} &\leq \left\| t^{-\frac{\alpha}{2}} \int_t^1 \left(\frac{t}{s}\right)^{\frac{k}{2}} \|Q_s f\|_{L^2} \frac{\mathrm{d}s}{s} \right\|_{L^2(t^{-1}\mathrm{d}t)} \\ &\leq \frac{2}{k-\alpha} \|f\|_{\mathcal{H}^\alpha}. \end{aligned}$$

□

One can see that the bound diverges as α goes to 0 if the operator does not encode any cancellation, that is $k = 0$. In the case $\alpha = 0$, we have $\|P_t f\|_{L^\infty} \lesssim \|f\|_{L^\infty}$ hence the L^∞-bound holds. However the L^2-bound is not satisfied since $\|P_t f\|_{L^2} \lesssim \|f\|_{L^2}$ only implies

$$\int_0^1 \|P_t f\|_{L^2}^2 \frac{\mathrm{d}t}{t} \leq \|f\|_{L^2}^2 \int_0^1 \frac{\mathrm{d}t}{t} = \infty.$$

This will explain an important difference for paraproducts on negative Hölder and Sobolev spaces as one can see with Propositions 1.3 and 1.4.

1.3 – Intertwined paraproducts

We use the standard family of Gaussian operators to study the product of distributions as one can do using the Paley-Littlewood decomposition in the flat case; this leads to the definition of the paraproduct P and the resonant term $\mathsf{\Pi}$ to describe products. Then we introduce the paraproduct $\widetilde{\mathsf{P}}$ intertwined with P to describe solutions of PDEs.

1.3.1 – Paraproduct and resonant term

One can define the product of a distributions $f \in \mathcal{D}'(M)$ with a smooth function $g \in \mathcal{D}(M)$. If however the distribution f belongs to a Hölder space $\mathcal{C}^\alpha$ with $\alpha < 0$, one might hope to do better. It is indeed the case as we can see with the next Theorem which is nothing more than Young's integration condition.

Theorem. *The multiplication $(f,g) \mapsto fg$ extends as a unique bilinear operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha\wedge\beta}$ if and only if $\alpha + \beta > 0$.*

We are however interested in the case $\alpha + \beta < 0$ when dealing with singular stochastic PDEs, as we are interested to stochastic ODEs where Young's condition is not verified. Following [35], Bailleul, Bernicot and Frey in [6, 8, 7] have defined two bilinear operators $\mathsf{P}_f g$ and $\mathsf{\Pi}(f,g)$ such that we have the formal decomposition of the product of two distributions as

$$fg = \mathsf{P}_f g + \mathsf{\Pi}(f,g) + \mathsf{P}_g f$$

where the paraproducts $\mathsf{P}_f g$ and $\mathsf{P}_g f$ are well-defined for any distibutions $f,g \in \mathcal{D}'(M)$. Of course, this means that $\mathsf{\Pi}(f,g)$ does have a meaning for $f \in \mathcal{C}^\alpha$ and

$g \in \mathcal{C}^\beta$ if and only if $\alpha+\beta > 0$; this is the resonant term. We want this decomposition to keep an accurate track of the regularity of each terms. More precisely, $\mathsf{P}_f g$ and $\Pi(f,g)$ should belong to $\mathcal{C}^{\alpha+\beta}$ if $\alpha < 0$ while $\mathsf{P}_g f$ to the less regular space $\mathcal{C}^\alpha$ as it is the case for the torus. We construct in this work such paraproduct and resonant term for space distributions on our manifold M, we mainly follow [8] in the simpler spatial setting.

Let $f, g \in \mathcal{D}'(M)$. Formally, we have

$$\begin{aligned} fg &= \lim_{t\to 0} P_t^{(b)}\left(P_t^{(b)} f \cdot P_t^{(b)} g\right) \\ &= \int_0^1 \left\{Q_t^{(b)}\big(P_t^{(b)} f \cdot P_t^{(b)} g\big) + P_t^{(b)}\big(Q_t^{(b)} f \cdot P_t^{(b)} g\big) + P_t^{(b)}\big(P_t^{(b)} f \cdot Q_t^{(b)} g\big)\right\}\frac{\mathrm{d}t}{t} \\ &\qquad + P_1^{(b)}\left(P_1^{(b)} f \cdot P_1^{(b)} g\right). \end{aligned}$$

The last term being smooth, it does not bother us. Remark that the choice of the constant "1" is arbitrary and it might be useful to change it, as one can see with the construction of the Anderson Hamiltonian. The family $P^{(b)}$ does not encode any cancellation while $Q^{(b)}$ encodes cancellation of order $2b$ so each terms in the integral have one operator with a lot of cancellations and two with none. Since we do not have nice estimates for these terms, we want to transfer some of the cancellation from $Q^{(b)}$ to the $P^{(b)}$ in each term. To do so, we use the Leibnitz rule

$$V_i(fg) = V_i(f)g + fV_i(g).$$

For example, we have

$$\begin{aligned} \int_0^1 P_t^{(b)}\left((tV_i^2)Q_t^{(b-1)} f \cdot P_t^{(b)} g\right)\frac{\mathrm{d}t}{t} &= \int_0^1 P_t^{(b)}(\sqrt{t}V_i)\left((\sqrt{t}V_i)Q_t^{(b-1)} f \cdot P_t^{(b)} g\right)\frac{\mathrm{d}t}{t} \\ &\quad - \int_0^1 P_t^{(b)}\left((\sqrt{t}V_i)Q_t^{(b-1)} f \cdot (\sqrt{t}V_i)P_t^{(b)} g\right)\frac{\mathrm{d}t}{t} \end{aligned}$$

so if we denote by (c_1, c_2, c_3) the cancellation of the three operators in the integral, we have

$$(0, 2b, 0) = (1, 2b-1, 0) + (0, 2b-1, 1).$$

This shows that we will not be able to have cancellation for all three operators at the same time but at least two. This is where the notation $Q^\bullet$ comes into play and multiple uses of this trick allows to decompose the product as

$$fg = \sum_{\mathbf{a}\in\mathcal{A}_b} \sum_{\mathbf{Q}\in\mathsf{StGC}^{\mathbf{a}}} b_{\mathbf{Q}} \int_0^1 Q_t^{1\bullet}\left(Q_t^2 f \cdot Q_t^3 g\right)\frac{\mathrm{d}t}{t}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3)$, $\mathsf{StGC}^{\mathbf{a}} = \mathsf{StGC}^{a_1} \times \mathsf{StGC}^{a_2} \times \mathsf{StGC}^{a_3}$,

$$\mathcal{A}_b = \left\{(a_1, a_2, a_3) \in \mathbb{N}^3 \ ; \ a_1 + a_2 + a_3 = 2b \quad \text{and} \quad a_1, a_2 \text{ or } a_3 = b\right\}$$

and $b_{\mathbf{Q}} \in \mathbb{R}$ is a real coefficient associated to $\mathbf{Q}$. In particular, only one of the a_i in $\mathbf{a} \in \mathscr{A}_b$ can be less than $\frac{b}{2}$ and this gives us three terms $\mathsf{P}_f g, \mathsf{P}_g f$ and $\Pi(f,g)$ such that

$$fg = \mathsf{P}_f g + \Pi(f,g) + \mathsf{P}_g f + P_1^{(b)}\left(P_1^{(b)} f \cdot P_1^{(b)} g\right).$$

Definition. *Given two distributions $f, g \in \mathcal{D}'(M)$, we define the paraproduct and the resonant term as*

$$\mathsf{P}_f g := \sum_{\mathbf{a}\in\mathcal{A}_b; a_2<\frac{b}{2}} \sum_{\mathbf{Q}\in\mathsf{StGC}^{\mathbf{a}}} b_{\mathbf{Q}} \int_0^1 Q_t^{1\bullet}\left(Q_t^2 f \cdot Q_t^3 g\right)\frac{\mathrm{d}t}{t}.$$

and

$$\Pi(f,g) := \sum_{\mathbf{a}\in\mathcal{A}_b; a_2,a_3\geq\frac{b}{2}} \sum_{\mathbf{Q}\in\mathsf{StGC}^{\mathbf{a}}} b_{\mathbf{Q}} \int_0^1 Q_t^{1\bullet}\left(Q_t^2 f \cdot Q_t^3 g\right)\frac{\mathrm{d}t}{t}.$$

In particular, $\mathsf{P}_f g$ is a linear combination of

$$\int_0^1 Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}$$

and $\Pi(f,g)$ of

$$\int_0^1 P_t^{\bullet}\left(Q_t^1 f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$.

These operators enjoy the same continuity estimates as their Fourier counterparts from which one can recover Young's condition. We give the proof here as it is a good way to get used to the approximation theory.

Proposition 1.3. *Let $\alpha, \beta \in (-2b, 2b)$ be regularity exponents.*

- *If $\alpha \geq 0$, then $(f,g) \mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^\beta$.*
- *If $\alpha < 0$, then $(f,g) \mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*
- *If $\alpha + \beta > 0$, then $(f,g) \mapsto \Pi(f,g)$ is continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

Proof : Let us first consider the case $\alpha < 0$ and let $Q \in \mathsf{StGC}^r$ with $r > |\alpha+\beta|$. Recall that $\mathsf{P}_f g$ is a linear combination of terms of the form

$$\int_0^1 Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$. Since $\alpha < 0$, Proposition 1.2 gives

$$\begin{aligned}\left|\int_0^1 Q_s Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right| &\lesssim \int_0^1 \left(\frac{ts}{(t+s)^2}\right)^{\frac{r}{2}} \|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta} t^{\frac{\alpha+\beta}{2}}\frac{\mathrm{d}t}{t} \\ &\lesssim s^{\frac{\alpha+\beta}{2}}\|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta}\end{aligned}$$

for any $s \in (0,1)$ hence $\mathsf{P}_f g \in \mathcal{C}^{\alpha+\beta}$.

For $\alpha \geq 0$, we consider $Q \in \mathsf{StGC}^r$ with $r > |\beta|$. In this case, we have $|P_t f| \leq \|f\|_{\mathcal{C}^\alpha}$ for all $t \in (0,1)$ so

$$\left|\int_0^1 Q_s Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right| \lesssim s^{\frac{\beta}{2}}\|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta}$$

hence $\mathsf{P}_f g \in \mathcal{C}^\beta$.

For the resonant term, let $Q \in \mathsf{StGC}^r$ with $r > |\alpha+\beta|$. We have

$$\left|\int_0^1 Q_s P_t^\bullet \left(Q_t^1 f \cdot Q_t^2 g\right) \frac{\mathrm{d}t}{t}\right| \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \left(\int_0^s t^{\frac{\alpha+\beta}{2}} \frac{\mathrm{d}t}{t} + \int_s^1 \left(\frac{s}{t}\right)^{\frac{r}{2}} t^{\frac{\alpha+\beta}{2}} \frac{\mathrm{d}t}{t}\right)$$
$$\lesssim s^{\frac{\alpha+\beta}{2}} \|f\|_{\mathcal{C}^\alpha} \|f\|_{\mathcal{C}^\beta}$$

using that $\alpha+\beta > 0$ hence $\mathsf{\Pi}(f,g) \in \mathcal{C}^{\alpha+\beta}$.

□

We also have estimates for the Sobolev spaces whose proofs are given in Proposition A.5 from Appendix A.2.

Proposition 1.4. *Let $\alpha, \beta \in (-2b, 2b)$ be regularity exponents.*

- *If $\alpha > 0$, then $(f,g) \mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{H}^\beta$ to $\mathcal{H}^\beta$ and from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^\beta$.*
- *If $\alpha < 0$, then $(f,g) \mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{H}^\beta$ to $\mathcal{H}^{\alpha+\beta}$ and from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*
- *If $\alpha+\beta > 0$, then $(f,g) \mapsto \mathsf{\Pi}(f,g)$ is continuous from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*

In particular, this implies that $(f,g) \mapsto \mathsf{P}_f g$ is continuous from $L^2 \times \mathcal{C}^\beta$ to $\mathcal{H}^{\beta-\delta}$ for all $\delta > 0$. For Sobolev spaces, there is a small loss of regularity and one does not recover the space $\mathcal{H}^\beta$ while this does not happen for Hölder spaces. This comes from the remark following Proposition 1.2.

As in the works [38, 59] of Gubinelli, Ugurcan and Zachhuber, one last property of P and $\mathsf{\Pi}$ in terms of Sobolev spaces is that P is almost the adjoint of $\mathsf{\Pi}$ when L is self-adjoint in the sense that the difference is more regular. A careful track of the previous computation show that for all $\mathbf{a} \in \{(0,b,b),(b,0,b),(b,b,0)\}$ and $\mathbf{Q} \in \mathsf{StGC}^{\mathbf{a}}$, we have $b_{\mathbf{Q}} = 0$ except for

$$\mathbf{Q} = (P_t^{(b)}, Q_t^{(b/2)}, Q_t^{(b/2)}), (Q_t^{(b/2)}, P_t^{(b)}, Q_t^{(b/2)}), (Q_t^{(b/2)}, Q_t^{(b/2)}, P_t^{(b)})$$

where $b_{\mathbf{Q}} = 1$. Define the corrector for almost duality as

$$\mathsf{A}(a,b,c) := \big\langle a, \mathsf{\Pi}(b,c)\big\rangle - \big\langle \mathsf{P}_a b, c\big\rangle.$$

Proposition 1.5. *Assume L self-adjoint. Let $\alpha, \beta, \gamma \in (-2b, 2b)$ such that $\beta+\gamma < 1$ and $\alpha+\beta+\gamma \geq 0$. If $\alpha < 1$, then $(a,b,c) \mapsto \mathsf{A}(a,b,c)$ extends in a unique trilinear operator from $\mathcal{H}^\alpha \times \mathcal{C}^\beta \times \mathcal{H}^\gamma$ to $\mathbb{R}$.*

Proof : $\mathsf{A}(a,b,c)$ is a linear combination of

$$\int_0^1 \left\{\Big\langle a, P_t^{1\bullet}\big(Q_t^1 b \cdot Q_t^2 c\big)\Big\rangle - \Big\langle Q_t^{3\bullet}\big(P_t^2 a \cdot Q_t^4 b\big), c\Big\rangle\right\} \frac{\mathrm{d}t}{t}$$

with $P^1, P^2 \in \mathsf{StGC}^{[0,b]}$ and $Q^1, Q^2, Q^3, Q^4 \in \mathsf{StGC}^{\frac{b}{2}}$. We first consider $P^1, P^2 \in \mathsf{StGC}^0$. By construction of the paraproduct and the resonant term, we have $P^1 = P^2 = P^{(b)} =: P$ and $Q^1 = Q^2 = Q^3 = Q^4 = Q^{(b/2)} =: Q$ hence we consider

$$\int_0^1 \left\{\Big\langle a, P_t\big(Q_t b \cdot Q_t c\big)\Big\rangle - \Big\langle Q_t\big(P_t a \cdot Q_t b\big), c\Big\rangle\right\} \frac{\mathrm{d}t}{t}.$$

Since L is self-adjoint, P_t and Q_t are too and we have

$$\begin{aligned}\int_0^1 \Big\langle a, P_t\big(Q_t b\cdot Q_t c\big)\Big\rangle\frac{\mathrm{d}t}{t} &= \int_0^1 \Big\langle P_t a, Q_t b\cdot Q_t c\Big\rangle\frac{\mathrm{d}t}{t}\\ &= \int_0^1 \Big\langle P_t a\cdot Q_t b, Q_t c\Big\rangle\frac{\mathrm{d}t}{t}\\ &= \int_0^1 \Big\langle Q_t\big(P_t a\cdot Q_t b\big), c\Big\rangle\frac{\mathrm{d}t}{t}\end{aligned}$$

hence the difference is equal to 0. Let us now consider the terms with $P^1, P^2 \in \mathsf{StGC}^{[1,b]}$ and bound each of them independently. Since $\alpha+\beta+\gamma\geq 0$, we have

$$\begin{aligned}\left|\int_0^1 \Big\langle a, P_t^{1\bullet}\big(Q_t^2 b\cdot Q_t^3 c\big)\Big\rangle\frac{\mathrm{d}t}{t}\right| &\lesssim \|a\|_{\mathcal{H}^\alpha}\left\|\int_0^1 P_t^{1\bullet}\big(Q_t^2 b\cdot Q_t^3 c\big)\frac{\mathrm{d}t}{t}\right\|_{\mathcal{H}^{\beta+\gamma}}\\ &\lesssim \|a\|_{\mathcal{H}^\alpha}\|b\|_{\mathcal{C}^\beta}\|c\|_{\mathcal{H}^\gamma}\end{aligned}$$

with $\beta+\gamma<1$ and using $\alpha\in(0,1)$ we have

$$\begin{aligned}\left|\int_0^1 \Big\langle Q_t^{3\bullet}\big(P_t^2 a\cdot Q_t^4 b\big), c\Big\rangle\frac{\mathrm{d}t}{t}\right| &\lesssim \left\|\int_0^1 Q_t^{3\bullet}\big(P_t^2 a\cdot Q_t^4 b\big)\frac{\mathrm{d}t}{t}\right\|_{\mathcal{H}^{\alpha+\beta}}\|c\|_{\mathcal{H}^\gamma}\\ &\lesssim \|a\|_{\mathcal{H}^\alpha}\|b\|_{\mathcal{C}^\beta}\|c\|_{\mathcal{H}^\gamma}\end{aligned}$$

which completes the proof since $\alpha+\beta+\gamma\geq 0$.

□

1.3.2 – Intertwined paraproducts

The description of weak solution of PDEs involving a differential operator D using paracontrolled calculus necessitate to study how D and P interacte with each other. In the initial work [35] by Gubinelli, Imkeller and Perkowski, they consider the commutator between the paraproduct and the integral operator $\mathscr{L}^{-1}$. In the work [8], Bailleul, Bernicot and Frey introduced the new paraproduct $\widetilde{\mathsf{P}}$ intertwined with P via the relation

$$D\circ\widetilde{\mathsf{P}} = \mathsf{P}\circ D$$

with $D=\partial_t-\Delta$. This is a natural object to study weak formulation of PDEs involving products described by the paraproduct P. We only give the proofs for $D=\Delta$ since we only consider space paracontrolled calculus and refer to the work of Bailleul, Bernicot and Frey for $D=\partial_t-\Delta$. We want to define the new paraproduct $\widetilde{\mathsf{P}}$ intertwined with the paraproduct through

$$L\widetilde{\mathsf{P}}_f g = \mathsf{P}_f Lg.$$

Since L is not invertible, we use L^{-1} an inverse up to a smooth error term. Hence a more conceivable intertwining relation is

$$L\widetilde{\mathsf{P}}_f g = \mathsf{P}_f Lg - e^{-L}\left(\mathsf{P}_f Lg\right).$$

Definition. *Given any distributions $f, g \in \mathcal{D}'(M)$, we define $\widetilde{\mathsf{P}}_f g$ as*

$$\widetilde{\mathsf{P}}_f g := L^{-1}\mathsf{P}_f L g$$

for which we have the explicit formula

$$\widetilde{\mathsf{P}}_f g = \sum_{\mathbf{a}\in\mathscr{A}_b; a_2<\frac{b}{2}} \sum_{\mathbf{Q}\in\mathsf{StGC}^{\mathbf{a}}} b_{\mathbf{Q}} \int_0^1 \widetilde{Q}_t^{1\bullet}\left(Q_t^2 f\cdot \widetilde{Q}_t^3 g\right)\frac{\mathrm{d}t}{t}$$

where $\widetilde{Q}_t^1 := Q_t^1(tL)^{-1}$ and $\widetilde{Q}_t^3 := Q_t^3(tL)$.

It is immediate that $\widetilde{Q}^3$ belongs to StGC^{a_3+2}. The cancellation property of $\widetilde{Q}^1$ is given by the following Lemma. Remark that it is not in standard form anymore, this is where the GC class comes into play.

Lemma 1.6. *Let $Q \in \mathsf{StGC}^{\frac{b}{2}}$. Then $\widetilde{Q}_t := Q_t(tL)^{-1}$ defines a family that belongs to $\mathsf{GC}^{\frac{b}{2}-2}$ for b large enough.*

Proof : Since $Q \in \mathsf{StGC}^{\frac{b}{2}}$, there exist $I = (i_1, \dots, i_n), j \in \mathbb{N}$ and $c \in [\![1, b]\!]$ such that $\frac{b}{2} = |I| + j$ and

$$Q_t = (t^{\frac{|I|}{2}}V_I)(tL)^{\frac{j}{2}}P_t^{(c)}.$$

This immediatly follows from

$$\begin{aligned} Q_t(tL)^{-1} &= (t^{\frac{|I|}{2}}V_I)(tL)^{\frac{j}{2}}(tL)^{-1}P_t^{(c)} \\ &= (t^{\frac{|I|}{2}}V_I)(tL)^{\frac{j-2}{2}}P_t^{(c)}(\mathrm{Id} - e^L). \end{aligned}$$

□

This Lemma immediatly yields the following Proposition, that is $\widetilde{\mathsf{P}}$ has the same structure as P hence the same continuity estimates.

Proposition 1.7. *For any distribution $f, g \in \mathcal{D}'(M)$, $\widetilde{\mathsf{P}}_f g$ is given as a linear combination of terms of the form*

$$\int_0^1 \widetilde{Q}_t^{1\bullet}\left(Q_t^2 f\cdot \widetilde{Q}_t^3 g\right)\frac{\mathrm{d}t}{t}$$

where $\widetilde{Q}^1 \in \mathsf{GC}^{\frac{b}{2}-2}, Q^2 \in \mathsf{StGC}^{[0,b]}$ and $\widetilde{Q}^3 \in \mathsf{StGC}^{\frac{b}{2}+2}$. Thus for any regularity exponent $\alpha, \beta \in (-2b, 2b)$, we have the following continuity results.

- *If $\alpha \geq 0$, then $(f, g) \mapsto \widetilde{\mathsf{P}}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^\beta$.*
- *If $\alpha < 0$, then $(f, g) \mapsto \widetilde{\mathsf{P}}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

We also have the same associated Sobolev estimates.

- *If $\alpha > 0$, then $(f, g) \mapsto \widetilde{\mathsf{P}}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{H}^\beta$ to $\mathcal{H}^\beta$ and from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^\beta$.*
- *If $\alpha < 0$, then $(f, g) \mapsto \widetilde{\mathsf{P}}_f g$ is continuous from $\mathcal{C}^\alpha \times \mathcal{H}^\beta$ to $\mathcal{H}^{\alpha+\beta}$ and from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*

1.4 – Correctors and commutators

The study of PDEs with a singular product involves resonant terms given a function u paracontrolled by a noise-dependent function $X \in \mathcal{C}^\alpha$, that is

$$u = \widetilde{\mathsf{P}}_{u'} X + u^\sharp$$

with $u' \in \mathcal{C}^\alpha$ and $u^\sharp \in \mathcal{C}^{2\alpha}$ a smoother remainder. If $\alpha < 1$, the product $u\zeta$ is singular for $\zeta \in \mathcal{C}^{\alpha-2}$. However, we have the formal decomposition

$$\mathsf{\Pi}(u,\zeta) = \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{u'}X,\zeta\big) + \mathsf{\Pi}(u^\sharp,\zeta) = u'\mathsf{\Pi}(X,\zeta) + \mathsf{C}(u',X,\zeta) + \mathsf{\Pi}(u^\sharp,\zeta)$$

where the corrector C introduced by Gubinelli, Imkeller and Perkowski in [35] is defined as

$$\mathsf{C}(a_1,a_2,b) := \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1}a_2,b\big) - a_1\mathsf{\Pi}(a_2,b).$$

If $\frac{2}{3} < \alpha < 1$, then the product $\mathsf{\Pi}(u^\sharp,\zeta)$ is well-defined. Thus we are able to give a meaning to the product $u\zeta$ for u paracontrolled by X once we have a proper continuity estimate for C and a meaning to the product $X\zeta$; this is the rough paths philosophy. This last task is only a probabilistic one and does not impact the analytical resolution of the equation, this is the renormalisation step. We state here a continuity estimate for C while its proof is given in Proposition A.8 in Appendix A.2. The proof is based on the remark that for a fixed $x \in M$, we have

$$\begin{aligned} C\big(a_1,a_2,b\big)(x) &= \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1}a_2,b\big)(x) - a_1(x)\cdot\mathsf{\Pi}\big(a_2,b\big)(x) \\ &\simeq \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1-a_1(x)}a_2,b\big)(x) \end{aligned}$$

where $\simeq$ is equal up to a smooth term using that

$$a_1(x)\cdot a_2 \simeq a_1(x)\cdot\widetilde{\mathsf{P}}_1 a_2 = \widetilde{\mathsf{P}}_{a_1(x)}a_2.$$

Proposition 1.8. *Let $\alpha_1 \in (0,1)$ and $\alpha_2,\beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta < 0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta > 0,$$

then $(a_1,a_2,b) \mapsto \mathsf{C}(a_1,a_2,b)$ extends in a unique continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$.

We also have the following Proposition to work with Sobolev spaces.

Proposition 1.9. *Let $\alpha_1 \in (0,1)$ and $\alpha_2,\beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta < 0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta > 0,$$

then $(a_1,a_2,b) \mapsto \mathsf{C}(a_1,a_2,b)$ extends in a unique continuous operator from $\mathcal{H}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha_1+\alpha_2+\beta}$.

Note that a restriction appears here since the first parameter α_1 has to be smaller than 1. This is due to the fact that for any function $f \in \mathcal{C}^\alpha$ with $\alpha \geq 0$, one has

$$|f(x) - f(y)| \leq \|f\|_{\mathcal{C}^\alpha} d(x,y)^{\alpha\wedge 1}$$

with a factor not greater than 1 even for $\alpha > 1$. This means that we are not able to benefit from regularity greater than 1 only with a first order Taylor expansion.

To work with a function of regularity $\alpha_1 \in (1,2)$, one has to consider the refined corrector defined in the flat one dimensional case by

$$\mathsf{C}^{(1)}\big(a_1,a_2,b\big)(x) := \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1}a_2,b\big)(x) - a_1(x)\mathsf{\Pi}\big(a_2,b\big)(x) - a_1'(x)\mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{(x-\cdot)}a_2,b\big)(x)$$

that we interpret as a first order refined corrector for $x \in \mathbb{T}$. One could consider higher order refined correctors however it is not needed for the equations considered here. On a manifold M, the analogue is defined for any $x \in M$ by

$$\mathsf{C}_{(1)}\big(a,b,c\big)(x) := \mathsf{C}\big(a,b,c\big)(x) - \sum_{i=1}^{\ell} \gamma_i\big(V_i a\big)(x)\mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{\delta_i(x,\cdot)}b,c\big)(x)$$

where δ_i is given for $x,y \in M$ by

$$\delta_i(x,y) := \chi\big(d(x,y)\big)\langle V_i(x), \pi_{x,y}\rangle_{T_xM}$$

with χ a smooth non-negative function on $[0,+\infty)$ equal to 1 in a neighbourhood of 0 with $\chi(r)=0$ for $r \geq r_m$ the injectivity radius of the compact manifold M and $\pi_{x,y}$ a tangent vector of T_xM of length $d(x,y)$, whose associated geodesic reaches y at time 1. The functions γ_i are defined from the identity

$$\nabla f = \sum_{i=1}^{\ell} \gamma_i(V_i f)V_i,$$

for all smooth real-valued functions f on M.

Proposition 1.10. *Let $\alpha_1 \in (1,2)$ and $\alpha_2,\beta \in \mathbb{R}$. If*

$$\alpha_2+\beta<0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta>0,$$

then $(a_1,a_2,b) \mapsto \mathsf{C}^{(1)}(a_1,a_2,b)$ extends in a unique continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$.

The corrector C is needed to study ill-defined product, this is the condition $\alpha_2+\beta<0$. However, we also have to investigate well-defined product to get more accurate descriptions. For this purpose, we introduce the commutator

$$\mathsf{D}(a_1,a_2,b) := \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1}a_2,b) - \mathsf{P}_{a_1}\mathsf{\Pi}(a_2,b).$$

Proposition 1.11. *Let $\alpha_1 \in (0,1)$ and $\alpha_2,\beta \geq 0$. Then $(a_1,a_2,b) \mapsto \mathsf{D}(a_1,a_2,b)$ extends in a unique continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$ and from $\mathcal{H}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{H}^{\alpha_1+\alpha_2+\beta}$.*

Again, one can bypass the condition $\alpha_1 \in (0,1)$ using refined commutators. Note that in their initial work [35], Gubinelli, Imkeller and Perkowski call C a commutator whereas within the high order paracontrolled calculus of [7], the operator D is closer to be a commutator than C. We need two final operator. The commutator S that swaps paraproducts defined by

$$\mathsf{S}(a_1,a_2,b) := \mathsf{P}_b\widetilde{\mathsf{P}}_{a_1}a_2 - \mathsf{P}_{a_1}\mathsf{P}_b a_2$$

and the corrector R defined by

$$\mathsf{R}(a,b,c) := \mathsf{P}_a\widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab}c.$$

Proposition 1.12.

- *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta < 0$. Then $(a_1, a_2, b) \mapsto \mathsf{S}(a_1, a_2, b)$ extends in a unique continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$ and from $\mathcal{H}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{H}^{\alpha_1+\alpha_2+\beta}$.*

- *Let $\beta, \gamma \in \mathbb{R}$. Then $(a, b, c) \mapsto \mathsf{R}(a, b, c)$ extends in a unique continuous operator from $L^\infty \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^{\beta+\gamma}$.*

In addition to the refined versions of the corrector and the commutator, the high order paracontrolled calulus has to deal with iterated versions. Indeed, the term $\mathsf{C}(u', X, \zeta)$ is not defined if the noise is too rough hence u' also has to be paracontrolled. To deal with this, one needs to consider the iterated corrector

$$\mathsf{C}\big((a_1, a_2), b, c\big) := \mathsf{C}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b, c\big) - a_1 \mathsf{C}\big(a_2, b, c\big).$$

It satisfies the following continuity estimate.

Proposition 1.13. *Let $\alpha_1, \alpha_2 \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$. If*

$$\alpha_1 + \beta + \gamma < 0, \quad \alpha_2 + \beta + \gamma < 0 \quad \textit{and} \quad \alpha_1 + \alpha_2 + \beta + \gamma > 0,$$

then $(a_1, a_2, b, c) \mapsto \mathsf{C}\big((a_1, a_2), b, c\big)$ extends in a unique continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta+\gamma}$.

We do not state here all the continuity results here and refer to the different works [6, 7, 8, 11, 49, 50] for details. For example, one might need higher order iterated version such as

$$\mathsf{C}\big(\big((a_1, a_2), b\big), c, d\big) = \mathsf{C}\big((\widetilde{\mathsf{P}}_{a_1} a_2, b) c, d\big) - a_1 \mathsf{C}\big((a_2, b), c, d\big).$$

Depending on the problem under consideration, new correctors and commutators might be needed as one will see with quasilinear parabolic SSPDEs in 3, the random magnetic Laplacian in 5 or with the Brox diffusion in Chapter 7.

1.5 – Nonlinear paracontrolled expansion

In order to solve nonlinear PDEs with a fixed point formulation in a space of paracontrolled distributions, it is necessary to understand how it interact with a nonlinear expression. At first order, this is given by Bony's paralinearisation.

Proposition. *Let $\alpha \in (0, 1)$ and $f \in C^2(\mathbb{R})$. For any $u \in \mathcal{C}^\alpha$, we have*

$$\|f(u)^\sharp\|_{\mathcal{C}^{2\alpha}} \lesssim \|f\|_{C^2} (1 + \|u\|_{\mathcal{C}^\alpha})^2$$

with $f(u)^\sharp := f(u) - \mathsf{P}_{f'(u)} u$. If moreover $f \in C^3(\mathbb{R})$, then the map $u \mapsto f(u)^\sharp$ is locally Lipschitz and

$$\|f(u)^\sharp - f(v)^\sharp\|_{\mathcal{C}^{2\alpha}} \lesssim \|f\|_{C^3} (1 + \|u\|_{\mathcal{C}^\alpha} + \|v\|_{\mathcal{C}^\alpha})^2 \|u - v\|_{\mathcal{C}^\alpha}.$$

In general, one has the following Proposition.

Proposition 1.14. *Let $n \geq 0$ and $\alpha \in (0,1)$. For $f \in C^{n+1}(\mathbb{R})$ and $u \in \mathcal{C}^\alpha$, we have*

$$\left\| f(u) - \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{(-1)^i}{n!} \binom{k}{i} \mathsf{P}_{f^{(k)}(u)u^i} u^{k-i} \right\|_{\mathcal{C}^{(n+1)\alpha}} \lesssim \|f\|_{C^{n+1}} (1 + \|u\|_{\mathcal{C}^\alpha})^{n+1}.$$

If moreover $f \in C^{n+2}(\mathbb{R})$, we have

$$\|f(u)^\sharp - f(v)^\sharp\|_{\mathcal{C}^{(n+1)\alpha}} \lesssim \|f\|_{C^{n+2}} (1 + \|u\|_{\mathcal{C}^\alpha} + \|v\|_{\mathcal{C}^\alpha})^{n+1} \|u - v\|_{\mathcal{C}^\alpha}$$

where

$$f(u)^\sharp := f(u) - \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{(-1)^i}{n!} \binom{k}{i} \mathsf{P}_{f^{(k)}(u)u^i} u^{k-i}.$$

Proof : We have to prove that

$$f(u)^\sharp := f(u) - \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{(-1)^i}{n!} \binom{k}{i} \mathsf{P}_{f^{(k)}(u)u^i} u^{k-i}.$$

is a $(n+1)\alpha$-Hölder function. Using that $\mathsf{P}_1 f(u) = f(u)$ up to smooth term and that $\mathsf{P}_a b$ is the sum of terms of the form

$$\int_0^1 Q_t^{1\bullet}(Q_t^2 a \cdot P_t b) \frac{dt}{t}$$

with $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$, $f(u)^\sharp$ is a sum of terms of the form $\int_0^1 Q_t^{1\bullet}(r_t) \frac{dt}{t}$ with

$$r_t = Q_t^2\big(f(u)\big) - \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{(-1)^i}{n!} \binom{k}{i} Q_t^2\big(f(k)(u)u^i\big) P_t(u^{k-i}).$$

We need to get a bound on r_t in $L^\infty(M)$. We have for $x \in \mathcal{M}$

$$\begin{aligned} r_t(x) = \int_{M^2} & K_{Q_t^2}(x,y) K_{P_t}(x,z) \Big\{ f(u)(x) \\ & - \sum_{k=1}^{n} \sum_{i=0}^{k-1} \frac{(-1)^i}{n!} \binom{k}{i} \big(f(k)(u)u^i\big)(y) u^{k-i}(z) \Big\} \mu(\mathrm{d}y)\mu(\mathrm{d}z). \end{aligned}$$

Using a Taylor expansion for f, we have

$$\begin{aligned} r_t(x) &= \int_{[0,1]^{n+1}} f^{(n+1)}\Big(u(z) + \mathbf{s}_{n+1}\big(u(y) - u(z)\big)\Big) \mathbf{s}_n \big(u(y) - u(z)\big)^{n+1} \mathrm{d}\mathbf{s}_{n+1} \\ &\lesssim \|u\|_\alpha^{n+1} \, t^{\frac{(n+1)\alpha}{2}} \end{aligned}$$

with $\mathbf{s}_k = \prod_{i=1}^k s_i$ which allows us to conclude. The Lipschitz property follows from the same kind of computations.

$\square$

To solve (gPAM) in dimension 3 or (gKPZ) in dimension $1+1$, one only needs the third order expansion given by

$$\begin{aligned} f(u) = \mathsf{P}_{f'(u)}u &+ \frac{1}{2!}\left\{\mathsf{P}_{f^{(2)}(u)}u^2 - 2\mathsf{P}_{f^{(2)}(u)u}u\right\} \\ &+ \frac{1}{3!}\left\{\mathsf{P}_{f^{(3)}(u)}u^3 - 3\mathsf{P}_{f^{(3)}(u)u}u^2 + 3\mathsf{P}_{f^{(3)}(u)u^2}u\right\} + f(u)^\sharp. \end{aligned}$$

Using the correctors and commutators, this can be rewritten as

$$\begin{aligned} f(u) = \mathsf{P}_{f'(u)}u &+ \frac{1}{2!}\mathsf{P}_{f^{(2)}(u)}\mathsf{\Pi}(u,u) \\ &+ \frac{1}{3!}\mathsf{P}_{f^{(3)}(u)}\Big(8\mathsf{R}(u,u,u) + 2\mathsf{D}(u,u,u) + \mathsf{\Pi}\big(\mathsf{\Pi}(u,u),u\big)\Big) + f(u)^\sharp \end{aligned}$$

which can be interpreted as a "Taylor" expansion since each i-linear term if of regularity $i\alpha$ for $i \in \{1,2,3\}$. It is however not clear how to get such a general formula for arbitrary high order due to the large number of terms appearing in the computation.

1.6 – Generalisations and notations

The same theory can be adapted to deal with spacetime distribution in a parabolic scaling. Consider an horizon time $T>0$ and the parabolic manifold

$$\mathcal{M} := [0,T]\times M$$

equipped with the parabolic distance

$$\rho\big((\sigma,x),(\tau,y)\big) := \sqrt{|\tau-\sigma|} + d(x,y)$$

for $(\sigma,x),(\tau,y)\in\mathcal{M}$ and the parabolic measure

$$\nu := \mathrm{d}t\otimes\mu.$$

One can define analogously spacetime families $\mathsf{G}, \mathsf{StGC}^a$ and GC^a with

$$\mathcal{G}_t(e,e') = \frac{1}{\nu\big(B(e,\sqrt{t})\big)}\left(1 + c\frac{\rho(e,e')^2}{t}\right)^{-\ell}$$

for $e,e'\in\mathcal{M}$ and $t\in(0,1]$. While the heat semigroup is used to regularise in space, the regularisation in time is done through

$$\varphi^\star(f)(\tau) := \int_0^\infty \varphi(\tau-\sigma)f(\sigma)\mathrm{d}\sigma$$

with the scaling

$$\varphi_t(\cdot) := \frac{1}{t}\varphi\left(\frac{\cdot}{t}\right)$$

for $\varphi\in L^1(\mathbb{R})$. The standard family StGC^a of Gaussian operators with cancellation of order a is then

$$\Big((t^{\frac{|I|}{2}}V_I)(tL)^{\frac{j}{2}}P_t^{(c)}\otimes\varphi_t^\star\Big)_{t\in(0,1]}$$

where $a = |I| + j + 2k$, $c \in [\![1, b]\!]$ and φ a smooth function supported in $[2^{-1}, 2]$ with first derivative bounded by 1 such that

$$\int \tau^i \varphi(\tau) \mathrm{d}\tau = 0 \quad \text{for } 0 \leq i \leq k-1.$$

Using these families of spacetime operator can be used to construct analogue paraproducts $\mathsf{P}, \widetilde{\mathsf{P}}$ and resonant product $\mathsf{\Pi}$ in the parabolic spacetime setting of $\mathcal{M}$. In particular, the Besov spaces in this setting are in the parabolic scaling. In the Chapters 2 and 3, $\mathcal{C}$ will denote the parabolic Hölder spaces in which the equation are naturally solved. They have the same structure as their spatial counterparts thus everything work the same way even though the computations are more involved. This was introduced in [8] and extended to a higher order calculus in [7]. In this framework, one is interested in solving parabolic PDEs hence the natural intertwining relation is

$$(\partial_\tau + L) \circ \widetilde{\mathsf{P}} = \mathsf{P} \circ (\partial_\tau + L).$$

While there is a slightly larger lost of cancellation, $\widetilde{\mathsf{P}}$ still has the same structure as P thus satisfies the same continuity estimates. The particular case of space paracontrolled calculus was developped in [50] and is easier to understand at first.

The restriction of compactness on M is convenient since the space white noise does not belong to any unweighted function spaces on $\mathbb{R}^d$. Indeed, it does not satify any integrability properties at infinity. The technicalities of weight to deal with white noise on the full space was used by Hairer and Labbé in [40] and this was also present in the work on paracontrolled calculus with the heat semigroup by Bailleul, Bernicot and Frey [8]. This could be developped also in the spatial setting however we restrict ourselves to the case of bounded manifold M in this thesis for simplicity. While weights at infinity can be used to deal with the unbounded spatial setting, one can also use weight for small time to deal with rougher initial condition. Indeed, Chapters 2 and 3 work with smooth enough initial condition which is somehow not natural. This can be generalised using weight for small time since the divergence for the heat kernel as t goes to 0 can be explicitly computed, we refrain from doing so for simplicity but this could also be done.

Finally, we introduce a notation in order to deal with the large number of terms appearing in the study of parabolic PDEs such as (gPAM) equation in three dimensions or (gKPZ) equation in one dimension. Since we do not have a clear algebraic structure to order the expansion richer than just the scale of Hölder regularity, the only information we need to keep on terms is their regularity and the paracontrolled expansion rule. In the end, we only have two types of terms denoted by E and F. For example, one uses the corrector C to expand singular product as

$$\mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) = a_1 \mathsf{\Pi}(a_2, b) + \mathsf{C}(a_1, a_2, b)$$

while the commutator D has to be used if the product is well-defined with

$$\mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b\big) = \mathsf{P}_{a_1} \mathsf{\Pi}(a_2, b) + \mathsf{D}(a_1, a_2, b).$$

The first term correspond to the notation E while the second one to F. To be more precise, denote by $\mathsf{E}^\beta(\dots)$ given $\beta \in \mathbb{R}$ a generic multi-linear operator that sends formally $\mathcal{C}^\gamma$ to $\mathcal{C}^{\beta+\gamma}$ for any $\gamma \in \mathbb{R}$ and such that

$$\mathsf{E}^\beta(\widetilde{\mathsf{P}}_a b, \dots) = a \mathsf{E}^\beta(b, \dots) + \mathsf{E}^\beta(a, b, \dots)$$

for all $a \in \mathcal{C}^{|a|}, b \in \mathcal{C}^{|b|}$ with suitable regularity exponents $|a|$ and $|b|$. For F^β, the expansion rule is

$$\mathsf{F}^\beta(\widetilde{\mathsf{P}}_a b, \ldots) = \mathsf{P}_a \mathsf{F}^\beta(b, \ldots) + \mathsf{F}^\beta(a, b, \ldots).$$

Since we are only interested in expansion with respect to functionals of the unknown u, we write

$$\mathsf{E}^\beta(b, \ldots) = \mathsf{E}^{\beta+|b|}(\ldots)$$

and

$$\mathsf{F}^\beta(b, \ldots) = \mathsf{F}^{\beta+|b|}(\ldots)$$

when b only depends on the noise.

As explained, new correctors and commutators need to be introduced for different problem. One important example is to deal with derivatives for examples such as the KPZ equation. See Section 2.3 in Chapter 2 for a discussion on this. This will also be important for the random magnetic Laplacian and the Brox diffusion respectively in Chapters 5 and 7.

Chapter 2

Parabolic semilinear singular SPDEs

One of the simplest parabolic SSPDE is the stochastic heat equation with multiplicative noise
$$\partial_t u - \Delta u = u\zeta$$
with smooth initial condition $u_0 \in C^\infty$ and $\zeta \in \mathcal{C}^{\alpha-2}$ where $\alpha \in (0,1)$. A word of warning, we denote in this Chapter and the following by $\mathcal{C}^\beta$ the parabolic spacetime β-Hölder spaces. One of the most natural path to solve the equation is to perform a fixed point and take
$$u_1 := \mathscr{L}^{-1}(u_0\zeta).$$
Since u_0 is smooth, the product is well-defined with $u_0\zeta \in \mathcal{C}^{\alpha-2}$. Schauder estimates then imply $u_1 \in \mathcal{C}^\alpha$ and one would want to define
$$u_2 := \mathscr{L}^{-1}(u_1\zeta).$$
Since $\alpha < 1$, the product does not make sense hence one can not formulate the fixed point problem in $\mathcal{C}^\alpha$. This happens for the white noise in dimension $d > 1$ and corresponds to the singular nature of the equation. However the roughness of u_1 comes from ζ in a particular form dictated by the equation. This is where the controlled rough paths philosophy helps us and the paracontrolled calculus comes into play. We have the decomposition
$$u_1 = \mathscr{L}^{-1}\big(\mathsf{P}_{u_0}\zeta + \mathsf{P}_\zeta u_0 + \Pi(u_0,\zeta)\big) = \widetilde{\mathsf{P}}_{u_0}Z + u_1^\sharp$$
with $Z := \mathscr{L}^{-1}\zeta \in \mathcal{C}^\alpha$ and $u_1^\sharp \in \mathcal{C}^{2\alpha}$. Thus the singular product is formally given by
$$\Pi(u_1,\zeta) = \Pi\big(\widetilde{\mathsf{P}}_{u_0}Z + u_1^\sharp,\zeta\big) = u_0\Pi(Z,\zeta) + \mathsf{C}(u_0,Z,\zeta) + \Pi(u_1^\sharp,\zeta)$$
where the corrector $\mathsf{C}(u_0,Z,\zeta)$ is well-defined for $3\alpha - 2 > 0$. Hence for $\frac{2}{3} < \alpha < 1$, one only has to define $\Pi(Z,\zeta)$ which is independent of the equation as an element of its natural space $\mathcal{C}^{2\alpha-2}$, this is the renormalisation step. With this being done, the fixed point can be formulated in the solution space
$$\mathcal{D}(Z) := \Big\{\widetilde{\mathsf{P}}_{u'}Z + u^\sharp\ ;\ (u',u^\sharp) \in \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha}\Big\} \subset \mathcal{C}^\alpha.$$

To get local existence and uniqueness, one needs to show that for an horizon time small enough, the map

$$\Phi : \left| \begin{array}{ccc} \mathcal{D}(Z) & \to & \mathcal{D}(Z) \\ u & \mapsto & \mathscr{P}u_0 + \mathscr{L}^{-1}(u\xi) \end{array} \right.$$

is a well-defined contraction. Note that one needs to twist the domain by introducing a parameter $\beta \in (\frac{2}{3}, \alpha)$ and consider

$$\mathcal{D}^\beta(Z) := \left\{ \widetilde{\mathsf{P}}_{u'} Z + u^\sharp \ ; \ (u', u^\sharp) \in \mathcal{C}^\beta \times \mathcal{C}^{\alpha+\beta} \right\}$$

to get a contraction.

The limitation $\alpha \in (\frac{2}{3}, 1)$ appeared in the initial work [35] of Gubinelli, Imkeller and Perkowski and in the works [6, 8] of Bailleul, Bernicot and Frey on manifolds. This was extended with a higher order paracontrolled calculus in [7] by Bailleul and Bernicot to $\alpha \in (\frac{2}{5}, 1)$ as far as the fixed point formulation is concerned. The main obstacle to go all the way to the critical threshold $\alpha > 0$ is the understanding of the algebraic mechanism behind the large number of terms which appear in the computation of construct the solution space. Together with the development of a general renormalisation procedure, this is currently under investigation. The methods of higher order paracontrolled calculus can be briefly outlined as follows. Consider for example $\frac{1}{2} < \alpha < \frac{2}{3}$. The first order expansion

$$u = \widetilde{\mathsf{P}}_{u'} Z + u^\sharp \in \mathcal{D}(Z)$$

is not enough since the corrector $\mathsf{C}(u', Z_1, \zeta)$ and the products $u'\Pi(Z, \zeta)$ and $\Pi(u^\sharp, \zeta)$ are singular. This suggests that u' and $u^\sharp$ should be given by a first order paracontrolled expansion hence u should be given by a second order paracontrolled expansion of the form

$$u = \widetilde{\mathsf{P}}_{u_1} Z_1 + \widetilde{\mathsf{P}}_{u_2} Z_2 + u^\sharp \quad \text{and} \quad u_1 = \widetilde{\mathsf{P}}_{u_{11}} Z_1 + u_1^\sharp$$

with $u_1, u_2, u_{11} \in \mathcal{C}^\alpha$, $u_1^\sharp \in \mathcal{C}^{2\alpha}$ and $u^\sharp \in \mathcal{C}^{3\alpha}$. To define the singular corrector $\mathsf{C}(u_1, Z_1, \zeta)$, one uses the iterated corrector to get

$$\mathsf{C}(u_1, Z_1, \zeta) = \mathsf{C}\big(\widetilde{\mathsf{P}}_{u_{11}} Z_1 + u_1^\sharp, Z_1, \zeta\big) = u_{11}\mathsf{C}(Z_1, Z_1, \zeta) + \mathsf{C}\big((u_{11}, Z_1), Z_1, \zeta\big) + \mathsf{C}(u_1^\sharp, Z_1, \zeta)$$

where $\mathsf{C}(Z_1, Z_1, \zeta)$ is singular but only noise-dependent hence defined through the renormalisation step as an element of its natural space $\mathcal{C}^{3\alpha-2}$. The term $\mathsf{C}(u_1^\sharp, Z_1, \zeta)$ does not seem singular since $4\alpha - 2 > 0$ however $u_1^\sharp \in \mathcal{C}^{2\alpha}$ with $2\alpha \in (1, 2)$ and one can only gain regularity less than one for the first argument of the corrector. This is where the refined corrector $\mathsf{C}^{(1)}$ appears to get

$$\mathsf{C}\big(u_1^\sharp, Z_1, \zeta\big)(x) = \mathsf{C}^{(1)}\big(u_1^\sharp, Z_1, \zeta\big)(x) + (u_1^\sharp)'(x)\Pi\big((x - \cdot)Z_1, \zeta\big)(x).$$

The two singular products are dealt as the first one using the corrector with

$$\Pi\big(\widetilde{\mathsf{P}}_{u_2} Z_2, \zeta\big) = u_2\Pi(Z_2, \zeta) + \mathsf{C}(u_2, Z_2, \zeta)$$

and

$$\Pi\big(u_1, \Pi(Z_1, \zeta)\big) = u_{11}\Pi\big(Z_1, \Pi(Z_1, \zeta)\big) + \mathsf{C}\big(u_{11}, Z_1, \Pi(Z_1, \zeta)\big) + \Pi\big(u_1^\sharp, \Pi(Z_2, \zeta)\big)$$

where $\Pi(Z_2, \zeta)$ and $\Pi\big(Z_1, \Pi(Z_1, \zeta)\big)$ need to be define within the renormalisation step. For higher paracontrolled expansion, one needs estimates on a number of correctors/commutators, their refined version and higher order iterated. The subspace of $\mathcal{C}^\alpha$ on which the fixed point is performed is constructed based on the notion of paracontrolled systems. To obtain a contraction for small horizon time, one used the triangular character of the system with adapted regularity exponents for the control of the remainders just as one needs the parameter β for $\alpha \in (\frac{2}{3}, 1)$.

In this chapter, we give the tools to solve the generalised (PAM) equation

$$(\partial_t + L)u = f(u)\xi$$

on a three-dimensional manifold with ξ a space white noise. We also give the tools to solve the generalised (KPZ) equation

$$(\partial_t + L)u = f(u)\zeta + g(u)(\partial u)^2$$

with ζ a spacetime white noise in dimension $1+1$. This is based on the work [7].

2.1 – Paracontrolled systems

Let $n \in \mathbb{N}^*$ and $\mathcal{T}$ be a finite set of reference functions. Assume that $\mathcal{T}$ is the union

$$\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i \subset \mathcal{C}^\alpha$$

of finite sets $\mathcal{T}_1, \ldots, \mathcal{T}_n$ where $\mathcal{T}_i \subset \mathcal{C}^{i\alpha}$ for $1 \le i \le n$. For any $\tau \in \mathcal{T}$, we denote by $|\tau|$ its Hölder regularity and for any word $a = (\tau_1, \ldots, \tau_k)$ with letter in $\mathcal{T}$, we define its homogeneity as

$$|a| := |\tau_1| + \ldots + |\tau_k|.$$

A paracontrolled system by $\mathcal{T}$ at order n is naturally indexed by the set of words

$$\mathscr{A} := \Big\{ a = (\tau_1, \ldots, \tau_k) \ ; \ k \ge 0, |a| \le n\alpha \Big\}$$

where $k = 0$ correspond to the empty word $a = \emptyset$. Indeed, a second order paracontrolled system by (Z_1, Z_2) is given by

$$\begin{aligned} u &= \widetilde{\mathsf{P}}_{u_1} Z_1 + \widetilde{\mathsf{P}}_{u_2^\sharp} Z_2 + u^\sharp \\ u_1 &= \widetilde{\mathsf{P}}_{u_{11}^\sharp} Z_1 + u_1^\sharp \end{aligned}$$

where $u_a^\sharp \in \mathcal{C}^{3\alpha - |a|}$ for any $a \in \mathscr{A} = \{\emptyset, 1, 2, 11\}$ with $\mathcal{T} = \{1, 2\}$. This motivates the following definition of a paracontrolled system, where the role of the family $(\beta_a)_{a \in \mathscr{A}}$ is to get a contraction in the fixed point formulation. At first sight, the β_a's can be thought of as equal to α.

Definition 2.1. *Let $(\beta_a)_{a \in \mathscr{A}}$ be a family of positive real numbers. A system paracontrolled by $\mathcal{T}$ at order n is a family $\widehat{u} = (u_a)_{a \in \mathscr{A}}$ of functions such that for all $a \in \mathscr{A}$, one has*

$$u_a = \sum_{\tau \in \mathcal{T}; |a\tau| \le n\alpha} \widetilde{\mathsf{P}}_{u_{a\tau}} \tau + u_a^\sharp,$$

with $u_a^\sharp \in \mathcal{C}^{n\alpha + \beta_a - |a|}$.

As for rough paths, the solution space consists of enhanced data $\widehat{u} := (u_a)_{a\in\mathscr{A}}$ where $u = u_\emptyset$ is actually the stochastic process solution to the PDE. The data $\widehat{u}$ is equivalent to the data of the remainders $\widehat{u}^\sharp := (u_a^\sharp)_{a\in\mathscr{A}}$. Each function u_a can be interpreted as a high order Gubinelli's derivative and belongs a priori to $\mathcal{C}^\alpha$. The Hölder regularity of each remainder $u_a^\sharp$ is measured by the parameter β_a and the homogeneity of a. In particular, the larger the homogeneity of a is, the more regular the remainder $u_a^\sharp$ is.

Remark : *In regularity structures, the addition of an algebraic structure on $\mathcal{T}$ for the description of the enhanced data allows the resolution of a large class of subcritical singular SPDEs. In paracontrolled calculus, this has not been developped yet and would be interesting, a perspective under investigation is the notion of operads. Note that $\mathscr{A}$ is the truncated Hopf algebra of words with letters in $\mathcal{T}$ with the homogeneity as graduation.*

The set $\mathcal{T}$ will be implicitly defined with the fixed point formulation hence one needs to have a paracontrolled expression for the right hand side of the equation for a function u described by the enhanced data of an arbitrary paracontrolled system. This is the content of the following Theorem.

Theorem 2.2. *Let $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ be a third order paracontrolled system by a set $\mathcal{T}$. Then*

$$f(u)\zeta = \mathsf{P}_{f(u)}\zeta + \sum_{|a|\leq 2\alpha} \mathsf{P}_{f'(u)u_a}\zeta_a + \sum_{|ab|\leq 2\alpha} \mathsf{P}_{f^{(2)}(u)u_a u_b}\zeta_{a,b} + v^\sharp$$

where ζ_a and $\zeta_{a,b}$ are distributions that depend only on ξ and $\mathcal{T}$ of respective Hölder regularity $|a|+\alpha-2$ and $|ab|+\alpha-2$ and with $v^\sharp \in \mathcal{C}^{4\alpha-2}$ a remainder depending on $\widehat{u}$ and $\mathcal{T}$.

Proof : We have

$$f(u)\zeta = \mathsf{P}_{f(u)}\zeta + \mathsf{P}_\zeta f(u) + \Pi\big(f(u),\zeta\big).$$

Using the nonlinear paracontrolled expansion for $f(u)$, one has

$$\begin{aligned} f(u) &= \mathsf{P}_{f'(u)}u + \frac{1}{2!}\left\{\mathsf{P}_{f^{(2)}(u)}u^2 - 2\mathsf{P}_{f^{(2)}(u)u}u\right\} \\ &\quad + \frac{1}{3!}\left\{\mathsf{P}_{f^{(3)}(u)}u^3 - 3\mathsf{P}_{f^{(3)}(u)u}u^2 + 3\mathsf{P}_{f^{(3)}(u)u^2}u\right\} + f(u)^\sharp. \end{aligned}$$

Using the correctors and commutators, this gives

$$\begin{aligned} f(u) &= \mathsf{P}_{f'(u)}u + \frac{1}{2!}\mathsf{P}_{f^{(2)}(u)}\Pi(u,u) \\ &\quad + \frac{1}{3!}\mathsf{P}_{f^{(3)}(u)}\Big(8\mathsf{R}(u,u,u) + 2\mathsf{D}(u,u,u) + \Pi\big(\Pi(u,u),u\big)\Big) + f(u)^\sharp. \end{aligned}$$

This is where the notation E/F comes into play in order to simplify the computations. The previous equality rewrites as

$$f(u) = \mathsf{P}_{f'(u)}u + \frac{1}{2!}\mathsf{P}_{f^{(2)}(u)}\mathsf{E}(u,u) + \frac{1}{3!}\mathsf{P}_{f^{(3)}(u)}\Big(\mathsf{E}(u,u,u) + \mathsf{F}(u,u,u)\Big) + f(u)^\sharp.$$

Then the result follows from

$$\begin{aligned} \mathsf{E}^{-2}(u,u) &= \mathsf{E}^{-2}\big(\widetilde{\mathsf{P}}_{u_\tau}\tau, \widetilde{\mathsf{P}}_{u_\sigma}\sigma\big) \\ &= u_\tau u_\sigma \mathsf{E}^{-2+|\tau|+|\sigma|} + u_\tau u_{\sigma_1\sigma_2}\mathsf{E}^{-2+|\tau|+|\sigma_1|+|\sigma_2|} + u_\tau u_{\sigma_1\sigma_2\sigma_3}\mathsf{E}^{-2+|\tau|+|\sigma_1|+|\sigma_2|+|\sigma_3|} \\ &\quad + u_{\tau_1\tau_2}u_{\sigma_1\sigma_2}\mathsf{E}^{-2+|\tau_1|+|\tau_2|+|\sigma_1|+|\sigma_2|} + (5\alpha-2) \end{aligned}$$

and

$$\begin{aligned}\mathsf{E}^{-2}(u,u,u) &= \mathsf{E}^{-2}\big(\widetilde{\mathsf{P}}_{u_\tau}\tau, \widetilde{\mathsf{P}}_{u_\sigma}\sigma, \widetilde{\mathsf{P}}_{u_\gamma}\gamma\big) \\ &= u_\tau u_\sigma u_\gamma \mathsf{E}^{-2+|\tau|+|\sigma|+|\gamma|} + u_\tau u_\sigma u_{\gamma_1\gamma_2}\mathsf{E}^{-2+|\tau|+|\sigma|+|\gamma_1|+|\gamma_2|} + (5\alpha-2)\end{aligned}$$

obtained after multiple uses of the expansion rule for E. In particular, each time the expansion yields a paraderivatives of order more than 4, it goes in the remainder.

□

2.2 – Fixed point

As a guide for the fixed point formulation, we first detail the (PAM) equation in two dimensions where $\frac{2}{3} < \alpha < 1$ hence a first order paracontrolled expansion is enough as explained in the introduction of this chapter. A solution to the PDE

$$\mathscr{L}u = u\xi$$

with initial condition $u_0 \in \mathcal{C}^{2\alpha}$ is a fixed point of the map

$$u \mapsto \mathscr{L}^{-1}(u\xi) = \widetilde{\mathsf{P}}_u(\mathscr{L}^{-1}\xi) + \mathscr{L}^{-1}\big(\mathsf{P}_\xi u + \Pi(u,\xi)\big) + \mathcal{P}u_0$$

where $\mathcal{P}u_0 : (t,x) \mapsto \big(e^{-tL}u_0\big)(x)$ is the propagation of the initial condition. Since the equation is singular, this map is not well-defined from $\mathcal{C}^\alpha$ to $\mathcal{C}^\alpha$ and we consider

$$\Phi : (u', u^\sharp) \in \mathcal{D}^\beta(Z) \mapsto \Big(\ \widetilde{\mathsf{P}}_{u'}Z + u^\sharp\ ,\ R(u',u^\sharp)\ \Big) \in \mathcal{D}^\beta(Z)$$

with $Z = \mathscr{L}^{-1}\xi$ and

$$R(u',u^\sharp) := \mathscr{L}^{-1}\Big(\mathsf{P}_\xi u + u'\Pi(Z,\xi) + \mathsf{C}(u',Z,\xi) + \Pi(u^\sharp,\xi)\Big) + \mathcal{P}u_0$$

and where the solution space is given by

$$\mathcal{D}^\beta(Z) = \Big\{\widetilde{\mathsf{P}}_{u'}Z + u^\sharp\ ;\ (u',u^\sharp) \in \mathcal{C}^\beta \times \mathcal{C}^{\alpha+\beta} \text{ with } u'|_{t=0} = u_0 \text{ and } u^\sharp|_{t=0} = u_0\Big\}.$$

The natural norm on $\mathcal{D}^\beta(Z)$ is the product norm

$$\|u\|_{\mathcal{D}^\beta(Z)} := \|u'\|_{\mathcal{C}^\beta} + \|u^\sharp\|_{\mathcal{C}^{\alpha+\beta}}.$$

Note that the term $\mathcal{P}u_0$ goes in the remainder since $u_0 \in \mathcal{C}^{2\alpha}$, it is however possible to introduce a weight for small time to deal with $u_0 \in \mathcal{C}^\alpha$. To get local existence and uniqueness, we want to show that Φ is a contraction for an horizon time T small enough. This is where the parameter $\beta \in (\frac{2}{3},\alpha)$ comes into play with the bound

$$\|w\|_{\mathcal{C}^\gamma} \leq T^{\frac{\gamma-\gamma'}{2}}\|w\|_{\mathcal{C}^{\gamma'}}$$

for any spacetime function $w : [0,T]\times M \to \mathbb{R}$ equal to 0 at time $t = 0$ and regularity exponent $\gamma < \gamma'$. Indeed, this gives

$$\begin{aligned}\big\|\Phi(u',u^\sharp) - \Phi(v',v^\sharp)\big\|_{\mathcal{D}^\beta(Z)} &= \big\|\big(\widetilde{\mathsf{P}}_{u'}Z + u^\sharp\big) - \big(\widetilde{\mathsf{P}}_{v'}Z + v^\sharp\big)\big\|_{\mathcal{C}^\beta} + \big\|R(u',u^\sharp) - R(v',v^\sharp)\big\|_{\mathcal{C}^{\alpha+\beta}} \\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|\widetilde{\mathsf{P}}_{u'-v'}Z\|_{\mathcal{C}^\alpha} + T^{\frac{\alpha+\beta-\beta}{2}}\|u^\sharp - v^\sharp\|_{\mathcal{C}^{\alpha+\beta}} + T^{\frac{2\alpha-(\alpha+\beta)}{2}}\|R(u',u^\sharp) - R(v',v^\sharp)\|_{\mathcal{C}^{2\alpha}} \\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|u'-v'\|_{L^\infty}\|Z\|_{\mathcal{C}^\alpha} + T^{\frac{\alpha}{2}}\|u^\sharp - v^\sharp\|_{\mathcal{C}^{\alpha+\beta}} + T^{\frac{\alpha-\beta}{2}}\|R(u',u^\sharp) - R(v',v^\sharp)\|_{\mathcal{C}^{2\alpha}}\end{aligned}$$

for $(u',u^\sharp),(v',v^\sharp)\in\mathcal{D}^\beta(Z)$ hence Φ is a contraction for T small enough since $\beta<\alpha$. For higher order paracontrolled system, we choose the exponents $(\beta_a)_{a\in\mathscr{A}}$ such that $\beta_a>\beta_{a'}$ if the word a has more letters than a' or if a and a' have the same number of letters and $|a|>|a'|$. We consider the fixed point on the set of remainders $\widehat{u}^\sharp=(u_a^\sharp)_{a\in\mathscr{A}}$ and define the solution space as

$$\mathcal{S}(u_0):=\big\{\widehat{u}^\sharp;\ \forall a\in\mathscr{A},u_a\in\mathcal{C}^{3\alpha+\beta_a-|a|}\text{ and }u_a^\sharp|_{t=0}=h_a(\widehat{u})\big\}\subset\prod_{a\in\mathscr{A}}\mathcal{C}^{3\alpha+\beta_a-|a|}$$

where h_a given by the paracontrolled expansion of the right hand side defined below for $a\neq\emptyset$ and $h_\emptyset(u_0)=u_0$ with associated norm

$$\|\widehat{u}^\sharp\|_{\mathcal{S}}:=\sum_{a\in\mathscr{A}}\|u_a^\sharp\|_{\mathcal{C}^{3\alpha+\beta_a-|a|}}.$$

Given a family of remainders $\widehat{u}^\sharp\in\mathcal{S}(u_0)$, Theorem 2.2 gives a representation of the right hand side of the equation

$$\mathscr{L}^{-1}\big(f(u)\xi\big)=\sum_{\sigma\in\mathcal{T}'}\widetilde{\mathsf{P}}_{h_\sigma(\widehat{u})}\sigma+\mathscr{L}^{-1}\big(v^\sharp\big)$$

with $\mathcal{T}'$ a set of functions depending on ξ and $\mathcal{T}$. For example, the only term of homogeneity α in $\mathcal{T}'$ is $Z=\mathscr{L}^{-1}\xi$ with associated coefficient $h_Z(\widehat{u})=f(u)$. All the other terms $\sigma\in\mathcal{T}'$ are obtained from a word a or a couple of words (a,b) with respective homogeneity $|a|+\alpha$ or $|ab|+\alpha$ hence this yields the recursive construction of a concrete set $\mathcal{T}$ such that $\mathcal{T}=\mathcal{T}'$. The coefficients $h_\sigma(\widehat{u})$ is of the form $f(u)$, $f'(u)u_a$ or $f^{(2)}(u)u_au_b$ for some words $a,b\in\mathscr{A}$ and are also paracontrolled by $\mathcal{T}$. Thus the paracontrolled nonlinear expansion yields that $\mathscr{L}^{-1}\big(f(u)\xi\big)$ is described by a third order paracontrolled system by $\mathcal{T}$ with coefficient $h_a(\widehat{u})$ for $a\in\mathscr{A}$, we denote as $\Phi(\widehat{u}^\sharp)$ its family of remainders. By construction, the solution space $\mathcal{S}(u_0)$ is stable by the map Φ and a solution to the map equation is a fixed point of the map

$$\Phi:\mathcal{S}(u_0)\to\mathcal{S}(u_0).$$

We now prove that it is a contraction for T small enough.

Theorem 2.3. *For T small enough, the map Φ is a contraction from $\mathcal{S}(u_0)$ to itself.*

Proof: The contraction properties follows from the cascade of inequalities satisfied by the family $(\beta_a)_{a\in\mathscr{A}}$. Indeed, we want to control the norm of the difference

$$\|\Phi(\widehat{u})-\Phi(\widehat{v})\|_{\mathcal{S}}=:\|\widehat{w}\|_{\mathcal{S}}$$

for paracontrolled systems $\widehat{u},\widehat{v}\in\mathcal{S}(u_0)$. Since each $w_a^\sharp$ is equal to zero at time $t=0$, we want to make use of the bound

$$\|w_a^\sharp\|_{\mathcal{C}^\gamma}<T^{\frac{\gamma-\gamma'}{2}}\|w\|_{\mathcal{C}^{\gamma'}}$$

for any regularity exponent $\gamma<\gamma'$ as explained before. According to the decomposition of the right hand side of the equation, each terms $w_a^\sharp$ is given by a product of paraderivatives of $u-v$ of higher order than a in the sense that it implies for letter than the number of letters in a. In the first order expansion, this was the two bounds

$$\begin{aligned}\|w_\emptyset^\sharp\|_{\mathcal{C}^{\alpha+\beta}} &= \|R(u',u^\sharp)-R(v',v^\sharp)\|_{\mathcal{C}^{\alpha+\beta}}\\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|R(u',u^\sharp)-R(v',v^\sharp)\|_{\mathcal{C}^{2\alpha}}\\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|u'-v'\|_{\mathcal{C}^\beta}+T^{\frac{\alpha}{2}}\|u^\sharp-v^\sharp\|_{\mathcal{C}^{\alpha+\beta}}\end{aligned}$$

and

$$\begin{aligned}\|w_1^\sharp\|_{\mathcal{C}^\beta} &= \|\widetilde{\mathsf{P}}_{u'}Z+u^\sharp-(\widetilde{\mathsf{P}}_{v'}Z+v^\sharp)\|_{\mathcal{C}^\beta}\\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|\widetilde{\mathsf{P}}_{u'-v'}Z\|_{\mathcal{C}^\alpha}+T^{\frac{\alpha}{2}}\|u^\sharp-v^\sharp\|_{\mathcal{C}^{\alpha+\beta}}\\ &\lesssim T^{\frac{\alpha-\beta}{2}}\|u'-v'\|_{L^\infty}+T^{\frac{\alpha}{2}}\|u^\sharp-v^\sharp\|_{\mathcal{C}^{\alpha+\beta}}.\end{aligned}$$

In the general case, the paracontrolled expansion for the right hand side of the equation induce that any $a\in\mathscr{A}$ comes from $b\in\mathscr{A}$ or $b,c\in\mathscr{A}$ such that $|a|>|b|$ or $|a|>|bc|$. Thus we bound the associated remainder as

$$\begin{aligned}\|w_a^\sharp\|_{\mathcal{C}^{\beta_a}} &= \|f'(u)u_b-f'(v)v_b\|_{\mathcal{C}^{\beta_a}}\\ &\lesssim T^{\frac{\alpha-\beta_a}{2}}\|f'(u)-f'(v)\|_{\mathcal{C}^\alpha}\|u_b\|_{\mathcal{C}^{\beta_a}}+T^{\frac{\beta_b-\beta_a}{2}}\|f'(v)\|_{\mathcal{C}^\alpha}\|u_b-v_b\|_{\mathcal{C}^{\beta_b}}\\ &\lesssim T^{\frac{\alpha-\beta_a}{2}}\|u-v\|+T^{\frac{\beta_b-\beta_a}{2}}\|u_b-v_b\|_{\mathcal{C}^{\beta_b}}\\ &\lesssim \big(T^{\frac{\alpha-\beta_a}{2}}\|u_b\|_{\mathcal{C}^{\beta_a}}+T^{\frac{\beta_b-\beta_a}{2}}\|f'(v)\|_{\mathcal{C}^\alpha}\big)\|\widehat{u}-\widehat{v}\|\end{aligned}$$

for example in the case where a comes from b, the case where it comes from b,c holds with the same kind of computations. Thus we get a contraction for T small enough since $\beta_a>\beta_{a'}$ if the word a has more letters than a' or if a and a' have the same number of letters and $|a|>|a'|$. In particular, one has a finite number of terms of the form T^δ that needs to be small hence there exists an horizon time $T>0$ such that Φ is a contraction.

□

2.3 – Generalisations

To solve the (gKPZ) equation

$$\mathscr{L}u=f(u)\zeta+g(u)(\partial u)^2,$$

with ζ a spacetime white noise in dimension 1, the only missing ingredient is a paracontrolled expression for

$$(\partial u)^2=2\mathsf{P}_{\partial u}\partial u+\Pi(\partial u,\partial u)$$

when u is described by a paracontrolled system $\widehat{u}$. This can be done with the introduction of new correctors and commutators to deal with the first order differential operator ∂. For $u\in\mathcal{C}^\alpha$ with $\alpha<1$, the term $\mathsf{P}_{\partial u}\partial u$ is well-defined as an element of $\mathcal{C}^{2\alpha-2}$ since ∂u is only a distribution. This gives the hint that

$$\mathsf{P}_\partial\partial:\mathcal{C}^\alpha\times\mathcal{C}^\alpha\to\mathcal{C}^{2\alpha-2}$$

behaves more like the resonant term than the paraproduct as far as paracontrolled calculus is concerned. This is actually true and can be seen as follows. The paraproduct $\mathsf{P}_a b$ is given as a linear combination of terms of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t a\cdot\mathcal{Q}_t^2 b\right)\frac{\mathrm{d}t}{t}$$

for $\mathcal{Q}^1,\mathcal{Q}^2\in\mathsf{StGC}^{\frac{b}{2}}$ and $\mathcal{P}\in\mathsf{StGC}^{[0,b]}$. Thus $\mathsf{P}_{\partial a}\partial b$ is a linear combination of

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\widetilde{\mathcal{P}}_t a\cdot\widetilde{\mathcal{Q}}_t^2 b\right)\frac{\mathrm{d}t}{t^2}$$

where $\widetilde{\mathcal{P}}_t:=\sqrt{t}\partial\mathcal{P}_t\in\mathsf{StGC}^{[1,b+1]}$ and $\widetilde{\mathcal{Q}}_t^2:=\sqrt{t}\partial\mathcal{Q}_t^2\in\mathsf{StGC}^{\frac{b}{2}+1}$. Since the family $\widetilde{\mathcal{P}}$ encodes some cancellations, this is close to a resonant term with associated correctors

$$\begin{aligned}\mathsf{C}_\partial^{<}(a_1,a_2,b)&:=\mathsf{P}_{\partial\widetilde{\mathsf{P}}_{a_1}a_2}\partial b-a_1\mathsf{P}_{\partial a_2}\partial b,\\ \mathsf{C}_\partial^{>}(a,b_1,b_2)&:=\mathsf{P}_{\partial a}\partial\widetilde{\mathsf{P}}_{b_1}b_2-b_1\mathsf{P}_{\partial a}\partial b_2.\end{aligned}$$

The same holds for $\mathsf{\Pi}(\partial a,\partial b)$ with associated corrector

$$\mathsf{C}_\partial^{=}(a_1,a_2,b):=\mathsf{\Pi}\big(\partial\widetilde{\mathsf{P}}_{a_1}a_2,\partial b\big)-a_1\mathsf{\Pi}\big(\partial a_2,\partial b\big).$$

As for the corrector C, one can get continuity estimates for its refined and iterated versions which allows to get a paracontrolled expression for the term $(\partial u)^2$ when u is described by a paracontrolled system. This is an illustration of the flexibility of the paracontrolled calculus since one only needs to introduce new operators satisfying the correct continuity estimates to solve new problems. The same will be true in Chapters 3, 4 and 5 where the general idea and the basic tools will be the same with different operators. In particular, the class of what we call a paraproduct or a resonant term is very large. We only prove the estimates for some particular case in the Appendix, the proofs are similar in each cases.

While the paracontrolled calculus relies on classical tools from harmonic analysis, it still lacks a clear algebraic structure as it was developped for regularity structures. The works [9, 10] and [48] investigate the relation between the two approaches and it seems that while both have their advantages and disadvantages, the range of possible regularity should more or less be the same. Thus developping an algebraic framework as powerful as Hopf algebra are for regularity stuctures should be the only missing step to a formulation of all subcritical singular SPDEs through paracontrolled calculus.

In this Chapter, we only dealt with the analytical formulation of the equations as a fixed point and assumed that a number of ill-defined stochastic process were given to us. The construction of these is the probabilistic step in the resolution of SSPDEs and is called the renormalisation procedure. While this will be detailled in the different works hereafter, we do not give details for the semilinear and quasilinear parabolic SSPDEs of this Chapter and the next one. The main reason is that, as explained in the last paragraph, we do not have a clear algebraic framework to list all the singular processes that have to be renormalised. In the case of (gPAM) equation in dimension 3 and (gKPZ), one could do the details of the computations hidden in the E/F notation to get this list of more than one hundred terms appearing in the computations and renormalise by hand each singular ones. We do not think that this would be interesting to read for anyone thus we refrain us from doing so. In regularity structures, the algebraic formulation of this list of terms was one of the main goal of the work [18] by Bruned, Hairer and Zambotti while their renormalisation of the work [23] by Chandra and Hairer. In particular, our E/F notation can be interpreted as keeping only the leaf of their tree notation together with the loss of regularity since we forget the precise structure of each terms. In a sense, we smash the Hopf algebra of trees into the simpler structure of words. More importantly, the work of Chandra and Hairer is based on the BPHZ algorithm as in Bogoliubov, Parasiuk, Hepp and Zimmerman, coming from the renormalisation of Feynman diagrams in Quantum Field Theory and strongly relies on the euclidian structure through Taylor expansions. The extension of this method to the framework of manifolds is not trivial and was one of the goal of this thesis at first. It is still under investigation and is a very interesting question.

Chapter 3

Parabolic quasilinear singular SPDEs

In this Chapter, we explain how to solve the quasilinear version of any semilinear singular PDEs we are able to solve within the paracontrolled calculus thus including quasilinear generalised (KPZ) equation. This is done through two main arguments in addition to the method for semilinear PDEs.

(1) The introduction of new correctors and commutators to deal with first order and second order terms.

(2) Paracontrolled systems with a reference set
$$\mathcal{T} = \bigcup_{i=1}^{3} \mathcal{T}_i$$
where each $\mathcal{T}_i \subset \mathcal{C}^{i\alpha}$ is infinite.

While (1) is of interest in itself to consider other problems as we will see in other Chapters, (2) seems strongly related to the quasilinear character of the equations. Indeed, the infinite set $\mathcal{T}$ can be interpreted as the finite set obtained by the semilinear methods with "decorations" and this also appears in the other methods developped to solve such PDEs. A word of warning, we denote in this Chapter as in the previous one by $\mathcal{C}^\beta$ the parabolic spacetime β-Hölder spaces. Consider

$$\partial_t u - d(u)\Delta u = f(u,\xi)$$

with $f(u,\xi)$ a nonlinear term associated to a semilinear PDE one can solve with the paracontrolled calculus and $d : \mathbb{R} \to (0,+\infty)$ a smooth function taking values in a compact of $(0,+\infty)$. Given u_0 smooth enough for the product with ξ to be well defined, we first rewrite the equation to fit more in the framework of the method for semilinear PDEs as

$$\partial_t u - d(u_0)\Delta u = \big(d(u) - d(u_0)\big)\Delta u + f(u,\xi).$$

Thus we define
$$L := -\sum_{i=1}^{d} V_i^2 \quad \text{with} \quad V_i := \sqrt{d(u_0)}\partial_i,$$

and the equation rewrites

$$\mathscr{L}u := (\partial_t + L)u = \varepsilon(u,\cdot)Lu + \sum_{i=1}^{d} a_i(u,\cdot)V_i u + f(u,\xi)$$

with

$$\begin{aligned}\varepsilon(u,\cdot) &:= d(u_0)^{-1}\big(d(u_0) - d(u)\big),\\ a_i(u,\cdot) &:= \Big(d(u_0)^{-1}\big(d(u_0) - d(u)\big) - 1\Big)\frac{\partial_i\big(d(u_0)\big)}{2d(u_0)}.\end{aligned}$$

As notation, we write $\Delta = \partial_1^2 + \ldots + \partial_d^2$ but as in Chapter 1, the first order differential operator ∂_i's might be more general A_i's. Since we assumed that the semilinear equation can be solve with paracontrolled calculus, we already have a representation of $f(u,\xi)$ of the correct form and the aim is to get an expression of the type

$$\varepsilon(u,\cdot)Lu + \sum_{i=1}^{d} a_i(u,\cdot)V_i u = \sum_{\sigma} \mathsf{P}_{v_\sigma}\sigma.$$

Due to the quasilinear character of the equation, we are not able to find a finite set $\mathcal{T}$ to get stable paracontrolled systems. Considering an infinite set obtained from $\mathcal{T}$ such that it is stable by the operator $\mathscr{L}^{-1}L$, we are able to construct a stable solution space. The convergence of the infinite paracontrolled system is obtained using that $\varepsilon(u,\cdot)$ is small for a small horizon time since it a spacetime function null at $t = 0$ of positive Hölder regularity. This is based on the work [11].

3.1 – Additional correctors and commutators

The terms that appear are of the form

$$f(u)Du = \mathsf{P}_{f(u)}Du + \mathsf{P}_{Du}f(u) + \mathsf{\Pi}\big(f(u), Du\big)$$

with $f : \mathbb{R} \to \mathbb{R}$ a smooth function and D a differential operator of order $|D|$ constructed from the V_i's. As for the KPZ equation, the operators

$$(a,b) \mapsto \mathsf{P}_{Da}b,\ \ \mathsf{\Pi}\big(a, Db\big)$$

look like resonant terms since $(t^{\frac{|D|}{2}}D)\mathcal{Q}_t$ belongs to $\mathsf{StGC}^{r+|D|}$ for $\mathcal{Q} \in \mathsf{StGC}^r$. Thus the associated correctors

$$\begin{aligned}\mathsf{C}_D^{<}(a_1,a_2,b) &:= \mathsf{P}_{D\widetilde{\mathsf{P}}_{a_1}a_2}b - a_1\mathsf{P}_{Da_2}b,\\ \mathsf{C}_D^{=}(a_1,a_2,b) &:= \mathsf{\Pi}\big(D\widetilde{\mathsf{P}}_{a_1}a_2, b\big) - a_1\mathsf{\Pi}\big(Da_2, b\big),\end{aligned}$$

satisfies adapted continuity estimates as well as their refined and iterated versions. The term

$$(a,b) \mapsto \mathsf{P}_a Db$$

does not look like a resonant term but like a paraproduct. Indeed, it is a linear combination of terms of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t a \cdot \mathcal{Q}_t^2 Db\right)\frac{\mathrm{d}t}{t} = \int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t a \cdot \widetilde{\mathcal{Q}}_t^2 b\right)\frac{\mathrm{d}t}{t^{1+|D|}}$$

with $\widetilde{\mathcal{Q}}_t^2 = (t^{\frac{|D|}{2}} D)\mathcal{Q}_t^2 \in \mathsf{StGC}^{\frac{b}{2}+|D|}$ hence there are no cancellations in the lower term. The paraproduct term

$$\mathsf{P}_{f(u)} u = \mathsf{P}_{f(u)} \widetilde{\mathsf{P}}_{u_\tau} \tau$$

in the semilinear method is dealt with using the operator

$$\mathsf{R}(a,b,c) = \mathsf{P}_a \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} c$$

hence we introduce

$$\mathsf{R}_D(a,b,c) = \mathsf{P}_a D \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} D c.$$

To sum up, we introduce the correctors

$$\begin{aligned}
\mathsf{C}_V^{<}(a_1,a_2,b) &:= \mathsf{P}_{V\widetilde{\mathsf{P}}_{a_1 a_2}} b - a_1 \mathsf{P}_{V a_2} b,\\
\mathsf{C}_L^{<}(a_1,a_2,b) &:= \mathsf{P}_{L\widetilde{\mathsf{P}}_{a_1 a_2}} b - a_1 \mathsf{P}_{L a_2} b,\\
\mathsf{C}_V^{=}(a_1,a_2,b) &:= \Pi\big(V\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - a_1 \Pi\big(V a_2, b\big),\\
\mathsf{C}_L^{=}(a_1,a_2,b) &:= \Pi\big(L\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - a_1 \Pi\big(L a_2, b\big),
\end{aligned}$$

to deal with the terms

$$\mathsf{P}_{V_i u} a_i(u,\cdot), \mathsf{P}_{Lu}\varepsilon(u,\cdot), \Pi\big(V_i u, a_i(u,\cdot)\big), \Pi\big(Lu, \varepsilon(u,\cdot)\big).$$

We introduce the operators

$$\begin{aligned}
\mathsf{R}_V(a,b,c) &:= \mathsf{P}_a V \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} L c,\\
\mathsf{R}_L(a,b,c) &:= \mathsf{P}_a V \widetilde{\mathsf{P}}_b c - \mathsf{P}_{ab} L c,
\end{aligned}$$

to deal with the terms

$$\mathsf{P}_{a_i(u,\cdot)} V_i u, \mathsf{P}_{\varepsilon(u,\cdot)} L u.$$

Proposition 3.1.

- *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

  $$\alpha_2 + \beta - 1 < 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \beta - 1 > 0,$$

  then the operators $\mathsf{C}_V^{<}$ and $\mathsf{C}_V^{=}$ have natural extensions as continuous operators from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-1}$.

- *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

  $$\alpha_2 + \beta - 2 < 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \beta - 2 > 0,$$

  then the operators $\mathsf{C}_L^{<}$ and $\mathsf{C}_L^{=}$ have natural extensions as continuous operators from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.

- *Let $\beta \in (0,1)$ and $\gamma \in \mathbb{R}$. Then the operator R_V has a natural extension as a continuous operator from $L^\infty \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ to $\mathcal{C}^{\beta+\gamma-1}$.*

- *Let $\beta \in (0,1)$ and $\gamma \in \mathbb{R}$. Then the operator R_L has a natural extension as a continuous operator from $L^\infty \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ to $\mathcal{C}^{\beta+\gamma-2}$.*

This is not exaclty what is done in [11] where the operators

$$\mathsf{V}(a,b) := V\widetilde{\mathsf{P}}_a b - \mathsf{P}_a V b \simeq \mathsf{R}_V(1,a,b)$$

and

$$\mathsf{L}(a,b) := L\widetilde{\mathsf{P}}_a b - \mathsf{P}_a L b \simeq \mathsf{R}_L(1,a,b).$$

We prefer to use R_V and R_L here as it seems more natural in the line of Chapter 2, this is not important since they are equal up to a smooth term.

3.2 – Infinite paracontrolled systems

Due to the second order term on the right hand side of the equation, we need to consider paracontrolled system by a set

$$\mathcal{T} = \bigcup_{i=1}^{3} \mathcal{T}_i$$

where each $\mathcal{T}_i \subset \mathcal{C}^{i\alpha}$ is infinite. This is because the set $\mathcal{T}$ needs to be stable under the operator $\mathscr{L}^{-1}L$ which is continuous from $\mathcal{C}^\gamma$ to itself for any $\gamma \in \mathbb{R}$, as explained with the fixed point formulation. Recall that $\mathscr{A}$ is the set of words with letters in $\mathcal{T}$ of homogeneity less than $n\alpha$. We suppose that $\mathcal{T}$ is coutable, it will be the case in the following. In addition to the defintion of a paracontrolled system, an infinite paracontrolled system satifies a convergence condition with respect to the norm of a word defined as

$$\|a\| := \|\tau_1\|_{\mathcal{C}^{|\tau_1|}} \dots \|\tau_k\|_{\mathcal{C}^{|\tau_k|}}$$

for $a = \tau_1 \dots \tau_k \in \mathscr{A}$. As for semilinear PDEs, the family $(\beta_a)_{a\in\mathscr{A}}$ is a tool to get a contraction for a small horizon time and can be taught as equal to α at first sight. While it is infnite here, it will only take a finite number of values, this will be crucial to guarantee the existence of a positive horizon time $T > 0$ such that the fixed point map is a contraction.

Definition 3.2. *Let $(\beta_a)_{a\in\mathscr{A}}$ be a family of positive real numbers. A system paracontrolled by $\mathcal{T}$ at order n is a family $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ of functions such that for all $a \in \mathscr{A}$, one has*

$$u_a = \sum_{\tau\in\mathcal{T};|a\tau|\leq n\alpha} \widetilde{\mathsf{P}}_{u_{a\tau}}\tau + u_a^\sharp,$$

with $u_a^\sharp \in \mathcal{C}^{n\alpha+\beta_a-|a|}$ and

$$\|\widehat{u}\| = \sum_{a\in\mathscr{A}} \|u_a^\sharp\|_{\mathcal{C}^{n\alpha+\beta_a-|a|}}\|a\| < \infty.$$

Since the convergence condition is always satisfied for a finite set $\mathcal{T}$, this definition is coherent with the notion of finite paracontrolled system. Since the fixed point is performed on the space of remainders, the size of $\widehat{u}$ is measured through the family of remainders $(u_a^\sharp)_{a\in\mathscr{A}}$. Proposition 3.3 proves that the condition $\|\widehat{u}\| < \infty$ guarantees the convergence of the infinite sums appearing in the paracontrolled system. The β_a's will be chosen in the interval $(\frac{2}{5}, \alpha)$ in a particular way explained in Section 3.3. In particular, they verify $\beta_a > \beta_{a'}$ for any $a, a' \in \mathscr{A}$ with a' a word containing a as a subword. Note that each u_a with $|a| < n\alpha$ belongs to $\mathcal{C}^\alpha$, while the u_a with $|a| = n\alpha$ are elements of $\mathcal{C}^{\beta_a}$. Putting together all the contributions from $\mathcal{T}_i$ for $i \in \{1, \dots, n\}$, each u_a in a paracontrolled system with $|a| < n\alpha$ is in particular required to have an expansion of the form

$$u_a = (\alpha) + (2\alpha) + \dots + (n\alpha + \beta_a - |a|)$$

as will be proved in the following propostion. As in the finite case, a paracontrolled system is triangular, the bigger $|a|$ the lesser we expand u_a, and is actually determined by the family $\widehat{u} = (u_a^\sharp)_{a\in\mathscr{A}}$ of remainders and we rewrite the convergence condition in terms of the remainders only.

Proposition 3.3. *Let $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ be a system paracontrolled by $\mathcal{T}$ at order n. One has*

$$\sum_{a\in\mathscr{A}} \|u_a\|_{\mathcal{C}^{\beta_a}}\|a\| \lesssim \|\widehat{u}\|.$$

In particular, this implies

$$\forall a \in \mathscr{A}, \quad \|u_a\|_{\mathcal{C}^{\beta_a}} \lesssim \|\widehat{u}\|.$$

Proof : Let $a \in \mathscr{A}$. A finite induction gives

$$\begin{aligned}
\|u_a\|_{\mathcal{C}^{\beta_a}} &\lesssim \sum_{\tau\in\mathcal{T};|a\tau|\le n\alpha} \|u_{a\tau}\|_{\mathcal{C}^{\beta_a}}\|\tau\| + \|u_a^\sharp\|_{\mathcal{C}^{\beta_a}} \\
&\lesssim \sum_{\tau\in\mathcal{T};|a\tau|\le n\alpha} \|u_{a\tau}\|_{\mathcal{C}^{\beta_{a\tau}}}\|\tau\| + \|u_a^\sharp\|_{\mathcal{C}^{\beta_a}} \\
&\lesssim \sum_{b\in\mathscr{A};|ab|\le n\alpha} \|u_{ab}^\sharp\|_{\mathcal{C}^{\beta_{ab}}}\|b\|
\end{aligned}$$

since $\beta_a \ge \beta_{ab}$ for any $b \in \mathscr{A}$. Thus

$$\sum_{a\in\mathscr{A}} \|u_a\|_{\mathcal{C}^{\beta_a}}\|a\| \lesssim \sum_{a\in\mathscr{A}} \|u_a^\sharp\|_{\mathcal{C}^{\beta_a}}\|a\| \lesssim \|\widehat{u}\|.$$

□

As for semilinear PDEs, we need an expression for the right hand side as a paracontrolled system by a set $\mathcal{T}'$ for an arbitrary $\widehat{u} = (u_a)_{a\in\mathscr{A}}$. The difference is that the system is not strictly triangular due to the second order term. This can be seen for example in the mild formulation with the term

$$\mathscr{L}^{-1}\big(\mathsf{P}_{\varepsilon(u,\cdot)u_\tau} L\tau\big) = \widetilde{\mathsf{P}}_{\varepsilon(u,\cdot)u_\tau}\big(\mathscr{L}^{-1}L\tau\big).$$

While each term $\tau \in \mathcal{T}$ only generates terms of higher homogeneity in the semilinear case, it gives rise here to a term $(\mathscr{L}^{-1}L)\tau \in \mathcal{T}'$ of the same homogeneity. This causes the set $\mathcal{T}$ to be infinite since it has to be stable under the operator $\mathscr{L}^{-1}L$.

Theorem 3.4. *Let $\widehat{u} = (u_a)_{a\in\mathscr{A}}$ be a third order paracontrolled system by a set $\mathcal{T}$. Then*

$$\begin{aligned}
f(u)\xi + \varepsilon(u,\cdot)Lu + \sum_{i=1}^d a_i(u,\cdot)V_i u &= \mathsf{P}_{f(u)}\zeta + \sum_{|a|\le 2\alpha} \mathsf{P}_{f'(u)u_a}\zeta_a^{(1)} + \sum_{|ab|\le 2\alpha} \mathsf{P}_{f^{(2)}(u)u_a u_b}\zeta_{a,b}^{(1)} \\
&+ \sum_{\tau\in\mathcal{T}} \mathsf{P}_{\varepsilon(u,\cdot)u_\tau} L\tau + \sum_{|a|\le 3\alpha;a\notin\mathcal{T}} \mathsf{P}_{\varepsilon(u,\cdot)u_a}\zeta_a^{(2)} + \sum_{|ab|\le 3\alpha} \mathsf{P}_{d_0^{-1}d'(u)u_a u_b}\zeta_{a,b}^{(2)} \\
&+ \sum_{|abc|\le 3\alpha} \mathsf{P}_{d_0^{-1}d^{(2)}(u)u_a u_b u_c}\zeta_{a,b,c}^{(2)} + \sum_{i=1}^d \sum_{|\tau|=\alpha} \mathsf{P}_{a_i(u,\cdot)u_\tau}\zeta_{i,\tau} + v^\sharp
\end{aligned}$$

where $\zeta_e^{(1)}, \zeta_e^{(2)}, \zeta_{i,e}$ are distributions that depend only on ξ and $\mathcal{T}$ of respective Hölder regularity $|e| + \alpha - 2, |e| - 2, |e| + 1$ and with $v^\sharp \in \mathcal{C}^{4\alpha-2}$ a remainder depending on $\widehat{u}$ and $\mathcal{T}$.

Proof : This works as Theorem 2.2 in Chapter 2. The major difference is the presence of an infinite number of terms, this is however hidden in the recursive definition of $\mathcal{T}$. In particular, the terms $f(u)\xi$ gives rise to the terms of the form

$$\mathsf{P}_{f'(u)u_a}\zeta_a^{(1)} \quad \text{and} \quad \mathsf{P}_{f^{(2)}(u)u_a u_b}\zeta_{a,b}^{(1)}.$$

For the new terms

$$a_i(u,\cdot)V_i u \quad \text{and} \quad \mathsf{P}_{Lu}\varepsilon(u,\cdot) + \Pi\big(Lu, \varepsilon(u,\cdot)\big),$$

we respectively get $\zeta_{i,e}$ and $\zeta_e^{(2)}$ with e a tuple of words. The term

$$\mathsf{P}_{\varepsilon(u,\cdot)}Lu$$

is the concrete second order term in the right hand side that encode the quasilinear character of the equation. It gives the terms $L\tau$ for any $\tau \in \mathcal{T}$, this is why the space $\mathcal{T}$ needs to be stable under the operator $\mathscr{L}^{-1}L$. See [11] for the details of the computations where one needs to use the differents correctors C_D and commutators R_D for $D \in \{V_i, L\}$ introduced before. The fact that one still gets convergence series will be explained in the fixed point Theorem, it mainly follows from the fact that each iteration of the operator $\mathscr{L}^{-1}L$ comes with a factor $\varepsilon(u,\cdot)$ as one can see on the right hand side of the equation.

$\square$

3.3 – Fixed point

Except for the presence of infinite sums, the method is the same as for semilinear PDEs. The family $(\beta_a)_{a\in\mathscr{A}}$ also has to verify that $\beta_a > \beta_{a'}$ if a' has more letter than a. Furthermore, we ask that $\beta_a > \beta_{a'}$ if a and a' have the same number of letter but $|a| < |a'|$. Since the set of possible number of letter or homogeneity for words in $\mathscr{A}$ is finite, it is possible to choose such a family that takes a finite number of values. In particular, it verifies

$$\alpha > \beta_a > \beta_\emptyset > \frac{2}{5}$$

for any $a \neq \emptyset$. The fixed point is again considered on the set of remainders $\widehat{u}^\sharp = (u_a^\sharp)_{a\in\mathscr{A}}$ and we define the solution space as

$$\mathcal{S}(u_0) := \big\{\widehat{u}^\sharp;\ \forall a \in \mathscr{A}, u_a \in \mathcal{C}^{3\alpha+\beta_a-|a|} \text{ and } u_a^\sharp|_{t=0} = h_a(\widehat{u})\big\} \subset \prod_{a\in\mathscr{A}} \mathcal{C}^{3\alpha+\beta_a-|a|}$$

where $h_\emptyset(u_0) = u_0$ and h_a given by the paracontrolled expansion of the right hand side defined below as for the semilinear equation with associated norm

$$\|\widehat{u}^\sharp\|_{\mathcal{S}} := \sum_{a\in\mathscr{A}} \|u_a^\sharp\|_{\mathcal{C}^{3\alpha+\beta_a-|a|}}\|a\|.$$

Since the product is infinite, one has to guarantee that $\mathcal{S}(u_0)$ is indeed a closed subspace of the Banach space $\prod_{a\in\mathscr{A}} \mathcal{C}^{3\alpha+\beta_a-|a|}$. This is granted by the term $\|a\|$ in the norm of the solution space. In the case of a finite set $\mathcal{T}$, the two norms with or without this term are equivalent while this is not true for infinite set $\mathcal{T}$. Given a family of remainders $\widehat{u}^\sharp \in \mathcal{S}(u_0)$, Theorem 3.4 gives a representation of the right hand side of the equation as

$$\mathscr{L}^{-1}\Big(f(u)\xi + \varepsilon(u,\cdot)Lu + \sum_{i=1}^d a_i(u,\cdot)V_i u\Big) = \sum_{\sigma\in\mathcal{T}'} \widetilde{\mathsf{P}}_{h_\sigma(\widehat{u})}\sigma + \mathscr{L}^{-1}\big(v^\sharp\big)$$

with $\mathcal{T}'$ a set of functions depending on ξ and $\mathcal{T}$ and explicit coefficients $h_\sigma(\widehat{u})$. This gives the construction of a set $\mathcal{T}$ such that $\mathcal{T} = \mathcal{T}'$, necessary infinite since it has to be stable by the operator $\mathscr{L}^{-1}L$. Thus the paracontrolled nonlinear expansion yields that the right hand side is described by a third order paracontrolled system by $\mathcal{T}$, we denote as $\Phi(\widehat{u}^\sharp)$ its family of remainders. By construction, the solution space $\mathcal{S}(u_0)$ is stable by the map Φ and a solution to the map equation is a fixed point of the map

$$\Phi : \mathcal{S}(u_0) \to \mathcal{S}(u_0).$$

We now prove that it is a contraction for T small enough.

Theorem 3.5. *For T smalll enough, the map Φ is a contraction from $\mathcal{S}(u_0)$ to itself.*

Proof : The contraction for an horizon time $T > 0$ small enough follows again from the cascade relations satisfied by the β_a's. There is one new subtility due to the terms

$$\big(\mathscr{L}^{-1}L\big)\tau$$

for any $\tau \in \mathcal{T}$, this is dealt with by the condition $\beta_a > \beta_{a'}$ for $|a| < |a'|$ with the same number of letters. While the set $\mathscr{A}$ is infinite here, the required conditions on the β_a's allow to suppose that it is actually a finite family hence there exists an horizon time $T > 0$ such that Φ is a contraction. The only things that is different is the setting on infinite paracontrolled systems hence we have to prove that the convergence condition is stable under Φ. This is granted by the fact that for each tuple e of k words, the paraderivative with respect to ζ_e is the product of the k paraderivatives of the associated words. See the proof of Theorem 10 from [11] for details.

□

As explained at the end of the previous Chapter, the set $\mathcal{T}_{\text{semi}}$ is quite large for the example of (gPAM) equation in three dimensions and (gKPZ) equation in one dimension. For the quasilinear version of these equations, the infinite set $\mathcal{T}_{\text{quasi}}$ has the same structure as $\mathcal{T}_{\text{semi}}$ in the following sense. Since it has to be stable under the operator $\mathscr{L}^{-1}L$, each time you add a term of the form

$$\mathsf{M}(X_1,\dots,X_n)$$

with $X_1,\dots,X_n$ functionnals of the noise and M a n-linear operator, you have to also add

$$(\mathscr{L}^{-1}L)^{k_0}\mathsf{M}\big((\mathscr{L}^{-1}L)^{k_1}X_1,\dots,(\mathscr{L}^{-1}L)^{k_n}X_n\big)$$

for any integers $k_0,k_1,\dots,k_n \in \mathbb{N}$. If one represents the finite set $\mathcal{T}_{\text{semi}}$ with trees, this corresponds to adding integer decorations on edges thus $\mathcal{T}_{\text{quasi}}$ is equipped with a natural graduation $c(\tau)$ corresponding to the number of times the operator $\mathscr{L}^{-1}L$ appears on the tree $\tau \in \mathcal{T}_{\text{quasi}}$. This does not seems surprising since the other works dealing with such quasilinear equations also add decorations on trees, however not integers one but infinite dimensional because of their parametrix approach. As far as renormalisation is concerned, this imposed a condition of convergence where the norm of each tree should be controlled by a geometric factor, that is

$$\|\tau\|_{\mathcal{C}^{|\tau|}} \lesssim K^{c(\tau)}$$

with constant K a positive constant.

Chapter 4

The Anderson Hamiltonian

In this Chapter, we define and study the Anderson Hamiltonian

$$H := L + \xi$$

where $-L$ is the Laplace-Beltrami operator on a compact two-dimensional manifold M without boundary or with smooth boundary under Dirichlet conditions. To apply the construction of the first Chapter, one needs to have an Hörmander representation for L. This is possible in this case with a number of vector fields possibly greater than the dimension, see for example Section 4.2.1 from Stroock's book [57]. The random potential ξ is a spatial white noise and belongs almost surely to $\mathcal{C}^{\alpha-2}$ for any $\alpha < 1$. For a generic function $u \in L^2$, the product $u\xi$ is ill-defined hence one needs to find a proper domain for the operator. A natural method would be to take the closure of the subspace of smooth functions for the operator norm

$$\|u\|_{L^2} + \|Hu\|_{L^2}.$$

However this yields a trivial domain since Hu has the same regularity as the noise because of the product $u\xi$ for smooth u thus does not belong to L^2. The idea is to construct a random domain $\mathcal{D}_\Xi$ depending on an enhancement Ξ of the noise obtained through a renormalisation procedure. To do so, we use the paraproduct to decompose the product for $u \in \mathcal{H}^\alpha$ as

$$u\xi = \mathsf{P}_u\xi + \mathsf{P}_\xi u + \Pi(u,\xi).$$

In this expression, the roughest term is $\mathsf{P}_u\xi \in \mathcal{C}^{\alpha-2}$ while $\mathsf{P}_\xi u + \Pi(u,\xi)$ formally belongs to $\mathcal{H}^{2\alpha-2}$. For a function u in the domain, we want to cancel out the roughest part of the product using the Laplacian term Lu, hence we want

$$Lu = \mathsf{P}_u\xi + v^\sharp$$

with $v^\sharp \in \mathcal{H}^{2\alpha-2}$. This suggests the paracontrolled expansion

$$u = \widetilde{\mathsf{P}}_u X + u^\sharp$$

with

$$X := -L^{-1}\xi$$

and $u^\sharp \in \mathcal{H}^{2\alpha}$. The operator L is not invertible but as for the definition of the intertwined paraproduct $\widetilde{\mathsf{P}}$ and L^{-1} denotes an inverse up to the regularising operator

e^{-L}, see Chapter 1. We insist that we want functions in the domain to encode exactly what is needed to have a cancellation between the Laplacian and the product. In particular, H is not treated at all like a perturbation of the Laplacian. At this point, two natural questions arise. Is the subspace of such paracontrolled functions dense in L^2 and can one make sense of the singular product?

1) For the first question, one can introduce a parameter $s > 0$, in the spirit of what Gubinelli, Ugurcan and Zachhuber did in [38], and consider the modified paracontrolled expansion
$$u = \widetilde{\mathsf{P}}^s_u X + u^\sharp_s$$
with the truncated paraproduct $\widetilde{\mathsf{P}}^s$ defined below. For $s = s(\Xi)$ small enough, the map $\Phi^s(u) := u - \widetilde{\mathsf{P}}^s_u X$ is invertible as a perturbation of the identity and one can show that the subspace of such paracontrolled functions is indeed dense. The parameter s will also be a very useful tool to investigate the different properties of H. Indeed, the Anderson operator will be given as
$$Hu = Lu^\sharp_s + F_{\Xi,s}(u)$$
with $F_{\Xi,s} : \mathcal{D}(H) \subset L^2 \to L^2$ an explicit operator and as s goes to 0, $u^\sharp_s$ gets closer to u while $F_{\Xi,s}$ diverges. These different representations of H will yield a family of bounds on the eigenvalues $\big(\lambda_n(\Xi)\big)_{n\geq 1}$ of H of the form
$$m^-(\Xi, s)\lambda_n - m(\Xi, s) \leq \lambda_n(\Xi) \leq m^+(\Xi, s)\lambda_n + m(\Xi, s)$$
with $(\lambda_n)_{n\geq 1}$ the eigenvalues of L. In partiular, $m^-(\Xi, s)$ and $m^+(\Xi, s)$ converge to 1 while $m(\Xi, s)$ diverges almost surely as s goes to 0.

2) For the second question, we use the corrector C with
$$\mathsf{\Pi}(u, \xi) = u\mathsf{\Pi}(X, \xi) + \mathsf{C}(u, X, \xi) + \mathsf{\Pi}(u^\sharp, \xi)$$
for u paracontrolled by X. One has to define the product $\mathsf{\Pi}(X, \xi)$ independently of the operator, this is the renormalisation step. To do so, we use the Wick product and set
$$\mathsf{\Pi}(X, \xi) := \lim_{\varepsilon\to 0} \Big(\mathsf{\Pi}(X_\varepsilon, \xi_\varepsilon) - \mathbb{E}\big[\mathsf{\Pi}(X_\varepsilon, \xi_\varepsilon)\big]\Big)$$
with ξ_ε a regularisation of the noise. In some sense explained in Proposition 5.8, the operator H is the limit of the renormalised operators
$$H_\varepsilon := L + \xi_\varepsilon - c_\varepsilon$$
with $c_\varepsilon := \mathbb{E}\big[\mathsf{\Pi}(X_\varepsilon, \xi_\varepsilon)\big]$ a smooth function diverging almost surely as ε goes to 0. Note that on the torus, the noise is invariant by translation and c_ε is constant.

The approach sketched above yields an operator $H : \mathcal{D}(H) \subset L^2 \to \mathcal{H}^{2\alpha-2}$ with $\mathcal{D}(H)$ the space of paracontrolled functions. In two dimensions, $2\alpha - 2 < 0$ hence one needs to refine the definition of the domain to get an unbounded operator in L^2. To this purpose, Allez and Chouk introduced in [2] the subspace of $\mathcal{D}(H)$ of strongly paracontrolled functions still dense in L^2. This was also used by Gubinelli, Ugurcan and Zachhuber in [38] and adapted to the dimension 3 using a Hopf-Cole type transformation. We present here a different approach based on a higher order expansion. In particular, the domain of H will consist of functions u such that

$$u = \widetilde{\mathsf{P}}_u X_1 + \widetilde{\mathsf{P}}_u X_2 + u^\sharp$$

where $X_1 \in \mathcal{C}^\alpha, X_2 \in \mathcal{C}^{2\alpha}$ are noise-dependent functions and $u^\sharp \in \mathcal{H}^2$. Note that since we want to get bounds with respect to the enhanced noise Ξ, quantitative estimates are needed and we keep track of the different explicit constants that appear, in particular how small s needs to be with respect to the noise. If one is only interest in qualitative results, details of almost all computations can be skipped.

The Anderson Hamiltonian is for example involved in the study of evolution equations such as the heat equation with random multiplicative noise

$$\partial_t u = \Delta u + u\xi$$

called the Parabolic Anderson model. It first appeared in [3] as a description of a physical phenomena involving quantum-mechanical motion with an effect of mass concentration called Anderson localization. It also describes random dynamics in random environment, see the book [42] of König for a complete survey in a discrete space setting. In dimension 1, the noise is regular enough for the multiplication to make sense and the operator has been constructed by Fukushima and Nakao in [34] without renormalisation using Dirichlet space methods. Dumaz and Labbé recently gave in [29] a very accurate asymptotic behaviors in one dimension of the Anderson localization. In two dimensions using paracontrolled calculus, Allez and Chouk were the first to construct the operator on the torus, see [2]. They introduced the space of strongly paracontrolled distributions to get an operator from L^2 to itself with a renormalisation procedure and proved self-adjointness with pure point spectrum. They gave bounds on its eigenvalues and a tail estimate for the largest one. They also studied the large volume limit and gave a bound on the rate of divergence. Then Labbé constructed the operator in dimension ≤ 3 in [45] with different boundary coundition using regularity structures. It relies on a reconstruction Theorem in Besov spaces from his work [40] with Hairer. He also proved self-adjointess with pure point spectrum and gave tail estimate for all the eigenvalues as well as bounds for the large volume limit. Chouk and van Zuijlen also studied the large volume limit in two dimensions, see [24]. Finally Gubinelli, Ugurcan and Zachhuber constructed in [38] the operator in dimension 2 and 3 on the torus using a different approach. With a finer description of the paracontrolled structure, they showed density of the domain in L^2 before studying the operator. They also proved self-adjointness with pure point spectrum considering the bilinear form associated to H and considered evolution PDEs associated to the Anderson Hamiltonian such as the Schrödinger equation or the wave equation. Zachhuber used this approach in [59] to prove Strichartz estimate in two dimensions.

We shall first construct in Section 4.1 the enhanced noise Ξ from ξ by a renormalisation procedure and prove exponential moments for its norm. The domain $\mathcal{D}_\Xi$ of H is constructed in Section 4.2 and proved to be dense using a truncated paraproduct $\widetilde{\mathsf{P}}^s$. We show in particular in Proposition 5.5 that the natural norms of $\mathcal{D}_\Xi$ are equivalent to the norm operator; this will give the upper bound for the eigenvalues. The Section is ended with the computation of the Hölder regularity of the elements of the domain. After showing that the operator is closed, we show in Section 4.3 that H is the limit of the operators H_ε in some sense which yields the symmetry of H. We then control in Proposition 5.7 the $\mathcal{H}^1$ norm of $u^\sharp$ from the associated bilinear form applied to u; this will give the lower bound for the eigenvalues. This gives self-adjointness and pure point spectrum using the Babuška-Lax-Milgram Theorem and we conclude the section with a bound on the convergence of the eigenvalues of H_ε to H.

As in the work of Allez and Chouk [2], Labbé [45] and of Gubinelli, Ugurcan and Zachhuber [38], we construct a dense random subspace of L^2 though a renormalisation step to get a self-adjoint operator with pure point spectrum. Our approach is different since we perform a second order expansion using paracontrolled calculus based on the heat semigroup on the manifold M. We refine the upper bounds on the eigenvalues obtained in [2] on the torus while also providing lower bounds. We get upper bounds for $\mathbb{P}(\lambda_n(\Xi) \leq \lambda)$ for λ to $+\infty$ and $-\infty$. For λ to $-\infty$, a bound was first given in [45] for a bounded domain with different boundary conditions. We have a more explicit dependence on n while a less precise bound with respect to λ. To the best of our knowledge, no bounds for λ to $+\infty$ were known. We also prove that the eigenfunctions of H belong to $\mathcal{C}^{1-}$ while the works [2, 45, 38] only gave Sobolev regularity. For the Schrödinger equation, the construction of H on a manifold yields immediatly the same result as Gubinelli, Ugurcan and Zachhuber get on the torus, see [38]. As in their work, our construction of the Hamilton Anderson on M could be used to study other evolution PDEs, this is done in the Chapter 6. All these results are new in our geometrical framework and come from [50].

In [59], Zachhuber proves Strichartz inequalities for the Schrödinger equation on the two-dimensional torus. In the joint work [51], we were able to extend the result to a two-dimensional manifold. Furthermore, we obtained Strichartz inequalities for the wave equation on a two-dimensional manifold using the almost sure Weyl-type law obtained for the Anderson Hamiltonian. This is the content of Sections 6.1 and 6.2 in Chapter 6.

4.1 – Renormalisation and enhanced noise

As explained in the introduction, an element of the domain of H should behave like the linear part $X := -L^{-1}\xi$ hence the product $u\xi$ does not make sense in two dimensions. Using the corrector, we are able to define the product $u\xi$ for u paracontrolled by X once the product $X\xi$ is defined. To do so, a naive approach would be to regularize the noise where $\xi_\varepsilon = \Psi(\varepsilon L)\xi$ is a regularisation of the noise and take ε to 0. The only condition we take is Ψ such that $(\Psi(\varepsilon L))_\varepsilon$ belongs to the class G, for example $\Psi(\varepsilon L) = e^{\varepsilon L}$ works. Since the product is ill-defined, the quantity $\Pi(X_\varepsilon, \xi_\varepsilon)$ diverges as ε goes to 0 with $X_\varepsilon := -L^{-1}\xi_\varepsilon$. The now usual way is to substract another diverging quantity c_ε such that the limit

$$\Pi(X, \xi) := \lim_{\varepsilon \to 0} \Big(\Pi(X_\varepsilon, \xi_\varepsilon) - c_\varepsilon \Big)$$

exists and take this as the definition of the product. This is the Wick renormalisation and the purpose of the following Theorem with the renormalised Anderson Hamiltonian

$$H_\varepsilon := L + \xi_\varepsilon - c_\varepsilon.$$

Theorem 4.1. *Let $\alpha < 1$ and*

$$c_\varepsilon := \mathbb{E}\Big[\Pi(X_\varepsilon, \xi_\varepsilon)\Big].$$

Then there exists a random distribution $\Pi(X,\xi)$ that belongs almost surely to $\mathcal{C}^{2\alpha-2}$ and such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\big\|\Pi(X,\xi)-(\Pi(X_\varepsilon,\xi_\varepsilon)-c_\varepsilon)\big\|^p_{\mathcal{C}^{2\alpha-2}}\Big]=0$$

for any $p\geq 1$.

Proof : Since the noise is Gaussian, we only need to control second order moment using hypercontractivity. The resonant term $\Pi(X_\varepsilon,\xi_\varepsilon)$ is a linear combination of terms of the form

$$I_\varepsilon := \int_0^1 P_t^\bullet\left(Q_t^1 X_\varepsilon\cdot Q_t^2\xi_\varepsilon\right)\frac{\mathrm{d}t}{t}$$

with $P\in\mathsf{StGC}^{[0,b]}$ and $Q^1,Q^2\in\mathsf{StGC}^{\frac{b}{2}}$. We also define the renormalised quantity

$$J_\varepsilon := I_\varepsilon-\mathbb{E}[I_\varepsilon].$$

Let $u\in(0,1), x\in M$ and $Q\in\mathsf{StGC}^r$ with $r>|2\alpha-2|$. The expectation $\mathbb{E}\left[|Q_u\big(I_\varepsilon\big)(x)|^2\right]$ is given by the integral over $M^2\times[0,1]^2$ of

$$K_{Q_uP_t^\bullet}(x,y)K_{Q_uP_s^\bullet}(x,z)\mathbb{E}\Big[Q_t^1X_\varepsilon(y)Q_t^2\xi_\varepsilon(y)Q_s^1X_\varepsilon(z)Q_s^2\xi_\varepsilon(z)\Big]$$

against the measure $\mu(\mathrm{d}y)\mu(\mathrm{d}z)(ts)^{-1}\mathrm{d}t\mathrm{d}s$. Using the Wick formula, we have

$$\begin{aligned}
\mathbb{E}\Big[Q_t^1X_\varepsilon(y)Q_t^2\xi_\varepsilon(y)Q_s^1X_\varepsilon(z)Q_s^2\xi_\varepsilon(z)\Big] &= \mathbb{E}\left[Q_t^1X_\varepsilon(y)Q_t^2\xi_\varepsilon(y)\right]\mathbb{E}\left[Q_s^1X_\varepsilon(z)Q_s^2\xi_\varepsilon(z)\right]\\
&\quad+\mathbb{E}\left[Q_t^1X_\varepsilon(y)Q_s^1X_\varepsilon(z)\right]\mathbb{E}\left[Q_t^2\xi_\varepsilon(y)Q_s^2\xi_\varepsilon(z)\right]+\mathbb{E}\left[Q_t^1X_\varepsilon(y)Q_s^2\xi_\varepsilon(z)\right]\mathbb{E}\left[Q_s^1X_\varepsilon(z)Q_t^2\xi_\varepsilon(y)\right]\\
&=(1)+(2)+(3)
\end{aligned}$$

and this yields

$$\mathbb{E}\left[|Q_u\big(I_\varepsilon\big)(x)|^2\right]=I_\varepsilon^{(1)}(x)+I_\varepsilon^{(2)}(x)+I_\varepsilon^{(3)}(x).$$

The first term corresponds exactly to the extracted diverging quantity since

$$I_\varepsilon^{(1)}=\mathbb{E}\left[\int_0^1 Q_uP_t^\bullet\left(Q_t^1X_\varepsilon\cdot Q_t^2\xi_\varepsilon\right)\frac{\mathrm{d}t}{t}\right]^2=\mathbb{E}\big[Q_u(I_\varepsilon)\big]^2$$

and we have

$$\mathbb{E}\left[|Q_u\big(J_\varepsilon\big)(x)|^2\right]=\mathbb{E}\left[\Big\{Q_u\big(I_\varepsilon\big)(x)-\mathbb{E}[Q_u\big(I_\varepsilon\big)](x)\Big\}^2\right]=I_\varepsilon^{(2)}(x)+I_\varepsilon^{(3)}(x).$$

Using that $(\Psi(\varepsilon L))_\varepsilon$ belongs to G, ξ is an isometry from L^2 to square-integrable random variables and Lemma 1.6, we have

$$\begin{aligned}
&I_\varepsilon^{(2)}(x)+I_\varepsilon^{(3)}(x)\\
&\quad\lesssim\int_{M^2}\int_{[0,1]^2}K_{Q_uP_t^\bullet}(x,y)K_{Q_uP_s^\bullet}(x,z)\big\langle\mathcal{G}_{2\varepsilon+t+s}(y,\cdot),\mathcal{G}_{2\varepsilon+t+s}(z,\cdot)\big\rangle^2\mu(\mathrm{d}y)\mu(\mathrm{d}z)ts\mathrm{d}t\mathrm{d}s\\
&\quad\lesssim\int_{M^2}\int_{[0,1]^2}K_{Q_uP_t^\bullet}(x,y)K_{Q_uP_s^\bullet}(x,z)\mathcal{G}_{2\varepsilon+t+s}(y,z)^2\mu(\mathrm{d}y)\mu(\mathrm{d}z)ts\mathrm{d}t\mathrm{d}s\\
&\quad\lesssim\int_{M^2}\int_{[0,1]^2}\mathcal{G}_{u+t}(x,y)\mathcal{G}_{u+s}(x,z)\mathcal{G}_{2\varepsilon+t+s}(y,z)^2\mu(\mathrm{d}y)\mu(\mathrm{d}z)ts\mathrm{d}t\mathrm{d}s\\
&\quad\lesssim\int_{M^2}\int_{[0,1]^2}(2\varepsilon+t+s)^{-\frac{d}{2}}\mathcal{G}_{u+t}(x,y)\mathcal{G}_{u+s}(x,z)\mathcal{G}_{2\varepsilon+t+s}(y,z)\mu(\mathrm{d}y)\mu(\mathrm{d}z)ts\mathrm{d}t\mathrm{d}s\\
&\quad\lesssim\int_{[0,1]^2}(2\varepsilon+t+s)^{-\frac{d}{2}}(\varepsilon+u+t+s)^{-\frac{d}{2}}ts\mathrm{d}t\mathrm{d}s\\
&\quad\lesssim(\varepsilon+u)^{2-d}
\end{aligned}$$

hence the family $\big(\sqcap(X_\varepsilon,\xi_\varepsilon)-c_\varepsilon\big)_{\varepsilon>0}$ is bounded in $\mathcal{C}^{2\alpha-2}$ for any $\alpha<1$ since $d=2$. These computations also show that the associated linear combination of

$$J := \int_0^1 \Big\{ P_t^\bullet\left(Q_t^1 X\cdot Q_t^2\xi\right) - \mathbb{E}\left[P_t^\bullet\left(Q_t^1X\cdot Q_t^2\xi\right)\right]\Big\}\frac{\mathrm{d}t}{t}$$

yields a well-defined random distribution of $\mathcal{C}^{2\alpha-2}$ for $\alpha<1$ that we denote $\sqcap(X,\xi)$. The same type of computations show the convergence and completes the proof.

$\square$

The enhanced noise is defined as

$$\Xi := \big(\xi,\sqcap(X,\xi)\big)\in\mathcal{X}^\alpha$$

where $\mathcal{X}^\alpha := \mathcal{C}^{\alpha-2}\times\mathcal{C}^{2\alpha-2}$. One has to keep in mind that the notation $\sqcap(X,\xi)$ is only suggestive. In particular for almost every ω, one has

$$\sqcap\big(X,\xi\big)(\omega)\neq\sqcap\big(X(\omega),\xi(\omega)\big)$$

since the product is almost surely ill-defined. We also denote the regularized enhanced noise $\Xi_\varepsilon := \big(\xi_\varepsilon,\sqcap(X_\varepsilon,\xi_\varepsilon)-c_\varepsilon\big)$ with the norm

$$\|\Xi-\Xi_\varepsilon\|_{\mathcal{X}^\alpha} := \|\xi-\xi_\varepsilon\|_{\mathcal{C}^{\alpha-2}} + \big\|\sqcap(X,\xi)-\sqcap(X_\varepsilon,\xi_\varepsilon)+c_\varepsilon\big\|_{\mathcal{C}^{2\alpha-2}}$$

which goes to 0 as ε goes to 0. Using that the noise is Gaussian and almost surely in $\mathcal{C}^{-1-\kappa}$ for all $\kappa>0$, we have exponential moment for the norm of the enhanced noise.

Proposition 4.2. *There exists $h>0$ such that*

$$\mathbb{E}\left[e^{h\|\xi\|^2_{\mathcal{C}^{\alpha-2}}+h\|\sqcap(X,\xi)\|_{\mathcal{C}^{2\alpha-2}}}\right]<\infty.$$

Proof : Let $t\in(0,1)$ and $Q\in\mathsf{StGC}^r$ with $r>|\alpha-2|$. Using the Gaussian hypercontractivity, we have

$$\begin{aligned}\mathbb{E}\left[\|Q_t\xi\|^p_{L^p_x}\right] &= \int_M \mathbb{E}\left[|Q_t\xi|^p(x)\right]\mu(\mathrm{d}x)\\ &\le (p-1)^{\frac{p}{2}}\int_M \mathbb{E}\left[|Q_t\xi|^2(x)\right]^{\frac{p}{2}}\mu(\mathrm{d}x)\end{aligned}$$

hence we only need to bound the second moment, which is bounded by

$$\mathbb{E}\left[|Q_t\xi|^2(x)\right] = \|K_{Q_t}(x,\cdot)\|^2_{L^2}\lesssim\frac{1}{\mu\big(B(x,\sqrt{t})\big)}.$$

Using that $\mathcal{B}^{\alpha-2+\frac{1}{p}}_{2p,2p}\hookrightarrow\mathcal{B}^{\alpha-2}_{\infty,\infty}$, we have

$$\begin{aligned}\mathbb{E}\left[e^{h\|\xi\|^2_{\mathcal{C}^{\alpha-2}}}\right] &= \sum_{p\ge0}\frac{h^p}{p!}\mathbb{E}\big[\|\xi\|^{2p}_{\mathcal{C}^{\alpha-2}}\big]\\ &\le\sum_{p=0}^{p_0}\frac{h^p}{p!}\mathbb{E}\big[\|\xi\|^{2p}_{\mathcal{C}^{\alpha-2}}\big]+\sum_{p>p_0}\frac{h^p}{p!}\mathbb{E}\left[\|\xi\|^{2p}_{\mathcal{B}^{\alpha-2+\frac{1}{p}}_{2p,2p}}\right]\\ &\lesssim\sum_{p=0}^{p_0}\frac{h^p}{p!}\mathbb{E}\big[\|\xi\|^{2p_0}_{\mathcal{C}^{\alpha-2}}\big]^{\frac{p}{p_0}}+\sum_{p>p_0}\frac{h^p(2p-1)^p}{p!}\mathrm{Vol}(M)\end{aligned}$$

for $p_0 > \frac{2}{1-\alpha}$ hence the result for h small enough. For the bound on $\mathsf{\Pi}(X,\xi)$, the computations are the same without the square since it belongs to the second Wiener chaos hence Gaussian hypercontractivity gives

$$\mathbb{E}\big[|Q_t\mathsf{\Pi}(X,\xi)|^p(x)\big] \leq (p-1)^p\ \mathbb{E}\big[|Q_t\mathsf{\Pi}(X,\xi)|^2(x)\big]^{\frac{p}{2}}.$$

□

4.2 – Domain of the Hamiltonian

We first motivate the definition of the domain. Let $\alpha \in (\frac{2}{3},1)$ such that ξ belongs almost surely to $\mathcal{C}^{\alpha-2}$. Let $X \in \mathcal{C}^\alpha$ be a noise-dependent function and consider $u = \widetilde{\mathsf{P}}_{u'}X + u^\sharp$ a function paracontrolled by X with $u' \in \mathcal{H}^\alpha$ and $u^\sharp \in \mathcal{H}^{2\alpha}$. Then

$$\begin{aligned} Hu &= Lu + \xi u \\ &= L\big(\widetilde{\mathsf{P}}_{u'}X + u^\sharp\big) + \mathsf{P}_u\xi + \mathsf{P}_\xi u + \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{u'}X + u^\sharp,\xi\big) \\ &= \mathsf{P}_{u'}LX + \mathsf{P}_u\xi + \Big(Lu^\sharp + \mathsf{P}_\xi u + u'\mathsf{\Pi}(X,\xi) + \mathsf{C}(u',X,\xi) + \mathsf{\Pi}(u^\sharp,\xi)\Big). \end{aligned}$$

Taking $u' = u$ and $-LX = \xi$, the first two terms cancel each other and we get

$$Hu = Lu^\sharp + \mathsf{P}_\xi u + u\mathsf{\Pi}(X,\xi) + \mathsf{C}(u,X,\xi) + \mathsf{\Pi}(u^\sharp,\xi) \in \mathcal{H}^{2\alpha-2}.$$

This yields an unbounded operator in L^2 with values in $\mathcal{H}^{2\alpha-2}$. Since $2\alpha - 2 < 0$, Hu does not belong to L^2 hence we do not have an operator from L^2 to itself and this makes harder to study the spectral properties of H. To get around this, Allez and Chouk introduced in [2] the subspace of functions u paracontrolled by $L^{-1}\xi$ such that Hu does belong to L^2 called strongly paracontrolled functions. This approach was also used by Gubinelli, Ugurcan and Zachhuber in [38] however we proceed differently and use higher order expansions. Let $X_1 := X$ and $X_2 \in \mathcal{C}^{2\alpha}$ be another noise-dependent function. Given $u_2 \in \mathcal{H}^\alpha$ and $u^\sharp \in \mathcal{H}^{3\alpha}$, we consider $u = \widetilde{\mathsf{P}}_u X_1 + \widetilde{\mathsf{P}}_{u_2}X_2 + u^\sharp$ and we have

$$\begin{aligned} Hu = {}& \mathsf{P}_{u_2}LX_2 + u\mathsf{\Pi}(X_1,\xi) + \mathsf{C}(u,X_1,\xi) + \mathsf{P}_{u_2}\mathsf{\Pi}(X_2,\xi) + \mathsf{D}(u_2,X_2,\xi) \\ &+ \mathsf{P}_u\mathsf{P}_\xi X_1 + \mathsf{S}(u,X_1,\xi) + \mathsf{P}_\xi\widetilde{\mathsf{P}}_{u_2}X_2 + \mathsf{P}_\xi u^\sharp + Lu^\sharp + \mathsf{\Pi}(u^\sharp,\xi). \end{aligned}$$

Taking $u_2 = u$ and $-LX_2 = \mathsf{\Pi}(X_1,\xi) + \mathsf{P}_\xi X_1$ cancels the terms of Sobolev regularity $2\alpha - 2$ and we get

$$\begin{aligned} Hu = {}& \mathsf{\Pi}\big(u,\mathsf{\Pi}(X_1,\xi)\big) + \mathsf{P}_{\mathsf{\Pi}(X_1,\xi)}u + \mathsf{C}(u,X_1,\xi) + \mathsf{P}_u\mathsf{\Pi}(X_2,\xi) + \mathsf{D}(u,X_2,\xi) \\ &+ \mathsf{S}(u,X_1,\xi) + \mathsf{P}_\xi\widetilde{\mathsf{P}}_u X_2 + \mathsf{P}_\xi u^\sharp + Lu^\sharp + \mathsf{\Pi}(u^\sharp,\xi) \end{aligned}$$

hence $Hu \in \mathcal{H}^{3\alpha-2} \subset L^2$. This motivates the following definition for the domain $\mathcal{D}_\Xi$ of H with

$$-LX_1 := \xi \quad \text{and} \quad -LX_2 := \mathsf{\Pi}(X_1,\xi) + \mathsf{P}_\xi X_1.$$

Definition. *We define the set $\mathcal{D}_\Xi$ of functions paracontrolled by Ξ as*

$$\mathcal{D}_\Xi := \big\{u \in L^2;\ u^\sharp := u - \widetilde{\mathsf{P}}_u X_1 - \widetilde{\mathsf{P}}_u X_2 \in \mathcal{H}^2\big\}.$$

The domain $\mathcal{D}_\Xi$ is the random subspace of functions $u \in L^2$ paracontrolled by X_1 and X_2 up to a remainder $u^\sharp \in \mathcal{H}^2$ given by the explicit formula

$$u^\sharp = \Phi(u) := u - \widetilde{\mathsf{P}}_u X_1 - \widetilde{\mathsf{P}}_u X_2.$$

With this notation, we have $\mathcal{D}_\Xi = \Phi^{-1}(\mathcal{H}^2)$ and since $X_1 + X_2 \in \mathcal{C}^\alpha$, we actually have $\mathcal{D}_\Xi \subset \mathcal{H}^\beta$ for every $\beta < \alpha$. However, we have no idea at this point if this domain is trivial or dense in L^2 and an inverse to Φ would be useful. However, it is not necessarily invertible so we introduce a parameter $s > 0$ and consider

$$\Phi^s : \left| \begin{array}{ccc} \mathcal{D}_\Xi & \to & \mathcal{H}^2 \\ u & \mapsto & u - \widetilde{\mathsf{P}}^s_u X_1 - \widetilde{\mathsf{P}}^s_u X_2 \end{array} \right.$$

where $\widetilde{\mathsf{P}}^s$ is defined as

$$\widetilde{\mathsf{P}}^s_f g := \sum_{\mathbf{a} \in \mathscr{A}_b; a_2 < \frac{b}{2}} \sum_{\mathbf{Q} \in \mathsf{StGC}^{\mathbf{a}}} b_{\mathbf{Q}} \int_0^s \widetilde{Q}^{1\bullet}_t \left(Q^2_t f \cdot \widetilde{Q}^3_t g \right) \frac{\mathrm{d}t}{t}.$$

The important property is that while still encoding the important information of the paraproduct $\widetilde{\mathsf{P}}$, the truncated paraproduct $\widetilde{\mathsf{P}}^s$ is small as an operator for s small; this is quantified as follows and proved in Proposition A.6 in Appendix.

Proposition 4.3. *Let $\gamma \in (0,1)$ be a regularity exponent and $X \in \mathcal{C}^\gamma$. For any $\beta \in [0,\gamma)$, we have*

$$\|u \mapsto \widetilde{\mathsf{P}}^s_u X\|_{L^2 \to \mathcal{H}^\beta} \lesssim \frac{s^{\frac{\gamma-\beta}{4}}}{\gamma - \beta} \|X\|_{\mathcal{C}^\gamma}$$

Since X_1 and X_2 depends continuously on Ξ, this implies the existence of $m > 0$ such that

$$\|\widetilde{\mathsf{P}}^s_u X_1 + \widetilde{\mathsf{P}}^s_u X_2\|_{\mathcal{H}^\beta} \le m \frac{s^{\frac{\alpha-\beta}{4}}}{\alpha - \beta} \|\Xi\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha}) \|u\|_{L^2}$$

thus the operator $u \mapsto \widetilde{\mathsf{P}}^s_u(X_1 + X_2)$ is continuous from L^2 to $\mathcal{H}^\beta$ for $\beta \in [0,\alpha)$ and arbitrary small as s goes to 0. Hence we get that

$$\Phi^s : \mathcal{H}^\beta \to \mathcal{H}^\beta$$

is invertible for $s = s(\Xi,\beta)$ small enough as a perturbation of the identity. Since $\widetilde{\mathsf{P}}_u X_i - \widetilde{\mathsf{P}}^s_u X_i$ is a smooth function for any $s > 0$, the domain is still given by

$$\mathcal{D}_\Xi = \Phi^{-1}(\mathcal{H}^2) = (\Phi^s)^{-1}(\mathcal{H}^2)$$

and we have a decomposition given by Φ^s for any $u \in \mathcal{D}_\Xi$, that is

$$u = \widetilde{\mathsf{P}}^s_u X_1 + \widetilde{\mathsf{P}}^s_u X_2 + \Phi^s(u).$$

In particular, we emphasize that **the domain does not depend on s** while the decomposition we consider for element of the domain might. We denote

$$x := \|\Xi\|_{\mathcal{X}^\alpha}$$

to keep track of the quantitative dependance with respect to the enhanced noise Ξ and lighten the notation. We use the letter x as a reminder of the noise-dependance. For any $0 \le \beta < \alpha$, we define

$$s_\beta(\Xi) := \left(\frac{\alpha - \beta}{mx(1+x)} \right)^{\frac{4}{\alpha-\beta}}$$

such that for $s < s_\beta(\Xi)$, the operator $\Phi^s : \mathcal{H}^\beta \to \mathcal{H}^\beta$ is invertible and we denote Γ its inverse. We choose to drop the parameter s in the notation to lighten the computations however the reader should keep in mind that the map Γ depends on s. It is implicitly characterized by the relation

$$\Gamma u^\sharp = \widetilde{\mathsf{P}}^s_{\Gamma u^\sharp} X_1 + \widetilde{\mathsf{P}}^s_{\Gamma u^\sharp} X_2 + u^\sharp$$

for any $u^\sharp \in \mathcal{H}^\beta$. Our choice of $\widetilde{\mathsf{P}}^s$ is motivated by the preservation of the intertwining relation

$$\widetilde{\mathsf{P}}^s = L^{-1} \circ \mathsf{P}^s \circ L$$

with P^s defined as $\widetilde{\mathsf{P}}^s$. The map Γ will be a crucial tool to study the domain $\mathcal{D}_\Xi$, in particular to show density in L^2. Continuity estimates for Φ^s and Γ are given in the next Proposition. Note that in the following, this bound of the form $\|a - b\| \le c$ will be used as $\|a\| \le \|b\| + c$ or $\|b\| \le \|a\| + c$.

Proposition 4.4. *Let $\beta \in [0, \alpha)$ and $s \in (0,1)$. We have*

$$\|\Phi^s(u) - u\|_{\mathcal{H}^\beta} \le \frac{m}{\alpha - \beta} s^{\frac{\alpha-\beta}{4}} x(1+x)\|u\|_{L^2}.$$

If moreover $s < s_\beta(\Xi)$, this implies

$$\|\Gamma u^\sharp\|_{\mathcal{H}^\beta} \le \frac{1}{1 - \frac{m}{\alpha-\beta} s^{\frac{\alpha-\beta}{4}} x(1+x)} \|u^\sharp\|_{\mathcal{H}^\beta}.$$

Proof : The bounds on Φ^s follows directly from Proposotion 4.3. Moreover since

$$\frac{m}{\alpha - \beta} s^{\frac{\alpha-\beta}{4}} x(1+x) < 1$$

for $s < s_\beta(\Xi)$, the map $\Phi^s : \mathcal{H}^\beta \to \mathcal{H}^\beta$ is invertible and we have

$$\|\Gamma u^\sharp\|_{\mathcal{H}^\beta} \le \frac{1}{1 - \frac{m}{\alpha-\beta} s^{\frac{\alpha-\beta}{4}} x(1+x)} \|u^\sharp\|_{\mathcal{H}^\beta}.$$

□

Let us insist that $\|u_s^\sharp\|_{\mathcal{H}^\beta}$ is always controlled by $\|u\|_{\mathcal{H}^\beta}$ while s need to be small depending for $\|u\|_{\mathcal{H}^\beta}$ to be controlled by $\|u_s^\sharp\|_{\mathcal{H}^\beta}$. We also define the map Γ_ε associated to the regularized noise Ξ_ε as

$$\Gamma_\varepsilon u^\sharp = \widetilde{\mathsf{P}}^s_{\Gamma_\varepsilon u^\sharp} X_1^{(\varepsilon)} + \widetilde{\mathsf{P}}^s_{\Gamma_\varepsilon u^\sharp} X_2^{(\varepsilon)} + u^\sharp$$

with

$$-LX_1^{(\varepsilon)} := \xi_\varepsilon \quad \text{and} \quad -LX_2^{(\varepsilon)} := \Pi(X_1^{(\varepsilon)}, \xi_\varepsilon) - c_\varepsilon + \mathsf{P}_{\xi_\varepsilon} X_1^{(\varepsilon)}.$$

It satisfies the same bound as Γ with $\|\Xi_\varepsilon\|_{\mathcal{X}^\alpha}$ and the following approximation Lemma holds. We do not need to explicit the constant, it depends polynomialy on the noise Ξ and diverges as s goes to $s_\beta(\Xi)$.

Lemma 4.5. *For any $0 \le \beta < \alpha$ and $0 < s < s_\beta(\Xi)$, we have*

$$\|\mathrm{Id} - \Gamma\Gamma_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\beta} \lesssim_{\Xi,s,\beta} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, this implies the norm convergence of Γ_ε to Γ with the bound

$$\|\Gamma - \Gamma_\varepsilon\|_{\mathcal{H}^\beta \to \mathcal{H}^\beta} \lesssim_{\Xi,s,\beta} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Proof : Given any $u \in \mathcal{H}^\beta$, we have $u = \Gamma\Gamma^{-1}(u) = \Gamma(u - \widetilde{\mathsf{P}}^s_u X_1 - \widetilde{\mathsf{P}}^s_u X_2)$. Using Proposition 4.4, we get

$$\begin{aligned}
\|u - \Gamma\Gamma_\varepsilon^{-1}(u)\|_{\mathcal{H}^\beta} &= \big\|\Gamma\big(u - \widetilde{\mathsf{P}}^s_u X_1 - \widetilde{\mathsf{P}}^s_u X_2\big) - \Gamma\big(u - \widetilde{\mathsf{P}}^s_u X_1^{(\varepsilon)} - \widetilde{\mathsf{P}}^s_u X_2^{(\varepsilon)}\big)\big\|_{\mathcal{H}^\beta} \\
&= \Big\|\Gamma\Big(\widetilde{\mathsf{P}}^s_u\big(X_1^{(\varepsilon)} - X_1\big) + \widetilde{\mathsf{P}}^s_u\big(X_2^{(\varepsilon)} - X_2\big)\Big)\Big\|_{\mathcal{H}^\beta} \\
&\le \frac{\alpha-\beta}{\alpha-\beta-ms^{\frac{\alpha-\beta}{4}}x(1+x)}\Big\|\widetilde{\mathsf{P}}^s_u\big(X_1^{(\varepsilon)} - X_1\big) + \widetilde{\mathsf{P}}^s_u\big(X_2^{(\varepsilon)} - X_2\big)\Big\|_{\mathcal{H}^\beta} \\
&\lesssim \frac{s^{\frac{\alpha-\beta}{4}}(1+x)}{\alpha-\beta-ms^{\frac{\alpha-\beta}{4}}x(1+x)}\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}\|u\|_{L^2}
\end{aligned}$$

using the Proposition 4.3 and that $X_i^{(\varepsilon)} - X_i$ is i-linear in $\Xi_\varepsilon - \Xi$ for $i \in \{1,2\}$. The second statement follows from

$$\|\Gamma_\varepsilon - \Gamma\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta} = \|\left(\mathrm{Id} - \Gamma\Gamma_\varepsilon^{-1}\right)\Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta} \le \|\mathrm{Id} - \Gamma\Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta}\|\Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta}$$

with the bound uniform in ε for $s < s_\beta(\Xi_\varepsilon)$

$$\|\Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta} \le \frac{\alpha-\beta}{\alpha-\beta-ms^{\frac{\alpha-\beta}{4}}x(1+x)}.$$

□

This allows to prove density of the domain.

Corollary. *The domain $\mathcal{D}_\Xi$ is dense in $\mathcal{H}^\beta$ for any $\beta \in [0,\alpha)$.*

Proof : Given $f \in \mathcal{H}^2$, $\Gamma(g_\varepsilon) \in \mathcal{D}_\Xi$ where $g_\varepsilon = \Gamma_\varepsilon^{-1}f \in \mathcal{H}^2$ thus we can conclude with the Lemma 4.5 that

$$\lim_{\varepsilon\to 0}\|f - \Gamma(g_\varepsilon)\|_{\mathcal{H}^\beta} = 0.$$

The density of $\mathcal{H}^2$ in $\mathcal{H}^\beta$ then yields the result.

□

Taking into account in the previous computation the smooth term e^{-L} coming from the intertwining relation, we are able to define H as an unbounded operator in L^2 with domain $\mathcal{D}_\Xi$ as follows.

Definition. *We define the Anderson Hamiltonian $H : \mathcal{D}_\Xi \to L^2$ as*

$$Hu = Lu^\sharp + \mathsf{P}_\xi u^\sharp + \mathsf{\Pi}(u^\sharp, \xi) + R(u)$$

with $u^\sharp = \Phi(u)$ and $R : \mathcal{D}_\Xi \to L^2$ given by

$$\begin{aligned}
R(u) := \mathsf{\Pi}\big(u, \mathsf{\Pi}(X_1,\xi)\big) &+ \mathsf{P}_{\mathsf{\Pi}(X_1,\xi)}u + \mathsf{C}(u, X_1, \xi) + \mathsf{P}_u\mathsf{\Pi}(X_2,\xi) + \mathsf{D}(u, X_2, \xi) \\
&+ \mathsf{S}(u, X_1, \xi) + \mathsf{P}_\xi\widetilde{\mathsf{P}}_u X_2 - e^{-L}\left(\mathsf{P}_u L X_1 + \mathsf{P}_u L X_2\right).
\end{aligned}$$

The parameter s does not appear in the definition of H, it is a tool to study the properties of the operator. Indeed, one has different representations of Hu as

$$Hu = Lu_s^\sharp + \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R(u) + \Psi^s(u)$$

where $u_s^\sharp := \Phi^s(u)$ and

$$\Psi^s(u) := \Big(L + \mathsf{P}_\xi \cdot + \mathsf{\Pi}(\cdot,\xi)\Big)\big(\widetilde{\mathsf{P}}_u^s - \widetilde{\mathsf{P}}_u\big)(X_1 + X_2).$$

The different representations of H through the parameter $s > 0$ will be useful to get different bounds. For example, we can compare the graph norm of H given as

$$\|u\|_H^2 := \|u\|_{L^2}^2 + \|Hu\|_{L^2}^2$$

and the natural norms of the domain

$$\|u\|_{\mathcal{D}_\Xi}^2 := \|u\|_{L^2}^2 + \|\Phi^s(u)\|_{\mathcal{H}^2}^2$$

with the following Proposition. For $s \in (0,1)$ and $\delta > 0$, we introduce the constant

$$m_\delta^2(\Xi, s) := k\left(s^{\frac{\alpha-2}{2}} x(1+x^2) + s^{\frac{\alpha-\beta}{4}} x^2(1+x^3) + \delta^{-3}\big(1 + s^{\frac{\alpha}{4}} x(1+x)\big)x^4(1+x^8)\right)$$

where the index "2" refers to $\mathcal{H}^2$ and for a constant $k > 0$ large enough depending only on M and L. In particular, it depends polynomialy on the enhanced noise and diverges as s or δ goes to 0.

Proposition 4.6. *Let $u \in \mathcal{D}_\Xi$ and $s > 0$. For any $\delta > 0$, we have*

$$(1-\delta)\|u_s^\sharp\|_{\mathcal{H}^2} \leq \|Hu\|_{L^2} + m_\delta^2(\Xi, s)\|u\|_{L^2}$$

and

$$\|Hu\|_{L^2} \leq (1+\delta)\|u_s^\sharp\|_{\mathcal{H}^2} + m_\delta^2(\Xi, s)\|u\|_{L^2}$$

with $u_s^\sharp = \Phi^s(u)$.

Proof: For any $s > 0$, we have

$$Hu = Lu_s^\sharp + \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R(u) + \Psi^s(u).$$

Then $Lu_s^\sharp \in L^2$ and for $\beta = \frac{1}{2}(\frac{2}{3} + \alpha)$, we have

$$\begin{aligned} \|R(u)\|_{L^2} &\lesssim x(1+x^2)\|u\|_{\mathcal{H}^\beta} \\ \|\Psi^s(u)\|_{L^2} &\lesssim s^{\frac{\alpha-2}{2}} x(1+x^2)\|u\|_{L^2} \\ \|\mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi)\|_{L^2} &\lesssim \|\xi\|_{\mathcal{C}^{\alpha-2}}\|u_s^\sharp\|_{\mathcal{H}^{\frac{4}{3}}}. \end{aligned}$$

One can bound the $\mathcal{H}^\beta$ norm of u using Proposition 4.4 with

$$\|u\|_{\mathcal{H}^\beta} \leq \|u_s^\sharp\|_{\mathcal{H}^\beta} + \frac{m}{\alpha-\beta} s^{\frac{\alpha-\beta}{4}} x(1+x)\|u\|_{L^2}$$

and since $\beta < 1$, one has

$$\|Lu_s^\sharp - Hu\|_{L^2} \lesssim \left(s^{\frac{\alpha-2}{2}} x(1+x^2) + s^{\frac{\alpha-\beta}{4}} x^2(1+x^3)\right)\|u\|_{L^2} + x(1+x^2)\|u_s^\sharp\|_{\mathcal{H}^{\frac{4}{3}}}.$$

Since $0 < \beta < 2$, we have for any $t > 0$

$$\begin{aligned} \|u_s^\sharp\|_{\mathcal{H}^{\frac{4}{3}}} &\lesssim \left\|\int_0^t (t'L)e^{-t'L} u_s^\sharp \frac{\mathrm{d}t'}{t'}\right\|_{\mathcal{H}^{\frac{4}{3}}} + \left\|e^{-tL} u_s^\sharp\right\|_{\mathcal{H}^{\frac{4}{3}}} \\ &\lesssim t^{\frac{2}{3}}\|u_s^\sharp\|_{\mathcal{H}^2} + t^{-\frac{4}{2}}\Big(1 + s^{\frac{\alpha}{4}} x(1+x)\Big)\|u\|_{L^2}. \end{aligned}$$

Take

$$t = \left(\frac{\delta}{kx(1+x^2)}\right)^{\frac{3}{2}}$$

with k the constant from the previous inequality and $\delta > 0$. This yields

$$\|Lu_s^\sharp - Hu\|_{L^2} \lesssim m_\delta^2(\Xi, s)\|u\|_{L^2} + \delta\|u_s^\sharp\|_{\mathcal{H}^2}.$$

and completes the proof.

$\square$

Finally, we can compute the Hölder regularity of the domain. In particular, this will implies the α-Hölder regularity of the eigenfunctions of H.

Proposition. *We have*

$$\mathcal{D}_\Xi \subset \mathcal{C}^\alpha.$$

Proof : The Besov embedding in two dimensions implies

$$\mathcal{H}^2 \hookrightarrow L^\infty$$

and $\Phi^s : L^\infty \to L^\infty$ is also invertible hence

$$\mathcal{D}_\Xi = \left(\Phi^s\right)^{-1}(\mathcal{H}^2) \subset L^\infty.$$

Given any $u \in \mathcal{D}_\Xi$, we get

$$\begin{aligned}\|u\|_{\mathcal{C}^\alpha} &\lesssim \|u\|_{L^\infty}\|X_1 + X_2\|_{\mathcal{C}^\alpha} + \|u_s^\sharp\|_{\mathcal{C}^\alpha} \\ &\lesssim_\Xi \|u\|_{L^\infty} + \|u_s^\sharp\|_{\mathcal{H}^2}\end{aligned}$$

and the proof is complete.

$\square$

4.3 – Self-adjointness and spectrum

4.3.1 – The operator is symmetric

We show that H is a closed self-adjoint operator on its dense domain $\mathcal{D}_\Xi \subset L^2$. This relies on approximation results and the Babuška-Lax-Milgram Theorem. The spectrum is pure point and the eigenvalues verify a min-max principle that allows to get estimates depending on the eigenvalues of L.

Proposition 4.7. *The operator H is closed on its domain $\mathcal{D}_\Xi$.*

Proof : Let $(u_n)_{n\geq 0} \subset \mathcal{D}_\Xi$ be a sequence such that

$$u_n \to u \quad \text{in } L^2 \quad \text{and} \quad Hu_n \to v \quad \text{in } L^2.$$

Proposition 4.6 gives that $\big(\Phi(u_n)\big)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}^2$ hence converges to $u^\sharp \in \mathcal{H}^2$. Since $\Phi : L^2 \to L^2$ is continuous, we have $\Phi(u) = u^\sharp$ hence $u \in \mathcal{D}_\Xi$. Finally, we have

$$\begin{aligned}\|Hu - v\|_{L^2} &\leq \|Hu - Hu_n\|_{L^2} + \|Hu_n - v\|_{L^2} \\ &\lesssim_\Xi \|u_n^\sharp - u^\sharp\|_{\mathcal{H}^2} + \|u - u_n\|_{L^2} + \|Hu_n - v\|_{L^2}\end{aligned}$$

hence $Hu = v$ and H is closed on $\mathcal{D}_\Xi$.

$\square$

In some sense, the operator H should be the limit of the renormalised H_ε as ε goes to 0. Since $\mathcal{D}(H_\varepsilon) = \mathcal{H}^2$, one can not compare directly the operators. However given any $u \in L^2$, we have

$$u = \big(\Gamma \circ \Phi^s\big)(u) = \lim_{\varepsilon\to 0} \big(\Gamma_\varepsilon \circ \Phi^s\big)(u).$$

Thus for $u \in \mathcal{D}_\Xi$, the approximation $u_\varepsilon := \big(\Gamma_\varepsilon \circ \Phi^s\big)(u)$ belongs to $\mathcal{H}^2$ and one can consider the difference

$$\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} = \|(H\Gamma - H_\varepsilon\Gamma_\varepsilon)u^\sharp\|_{L^2}$$

with $u^\sharp := \Phi^s(u)$. The following Proposition gives a bound for this quantity which yields the convergence as ε goes to 0 for s is small enough. We do not need to explicit the constant, it depends polynomialy on the enhanced noise Ξ and diverges as s goes to $s_0(\Xi)$.

Proposition 4.8. *Let $u \in \mathcal{D}_\Xi$ and $s > 0$ small enough. Then*

$$\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} \lesssim_{\Xi,s} \|u_s^\sharp\|_{\mathcal{H}^2}\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

with $u_s^\sharp = \Phi^s(u)$ and $u_\varepsilon := \Gamma_\varepsilon u_s^\sharp$. In particular, this implies that $H_\varepsilon\Gamma_\varepsilon$ converges to $H\Gamma$ in norm as ε goes to 0 as operators from $\mathcal{H}^2$ to L^2.

Proof : We have

$$H_\varepsilon u_\varepsilon = Lu_s^\sharp + \mathsf{P}_{\xi_\varepsilon} u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R_\varepsilon(u_\varepsilon) + \Psi_\varepsilon^s(u_\varepsilon)$$

where R_ε and Ψ_ε^s are defined as R and Ψ^s with Ξ_ε instead of Ξ. For $\beta = \frac{1}{2}(\frac{2}{3} + \alpha)$, we have

$$\begin{aligned}\|R(u) - R_\varepsilon(u_\varepsilon)\|_{L^2} &\leq \|R(u - u_\varepsilon)\|_{L^2} + \|(R - R_\varepsilon)(u_\varepsilon)\|_{L^2} \\ &\lesssim x(1 + x^2)\|u - u_\varepsilon\|_{\mathcal{H}^\beta} + (1 + x)\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}\|u_\varepsilon\|_{\mathcal{H}^\beta} \\ &\lesssim \Big(x(1 + x^2)\|\Gamma - \Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta} + (1 + x)\|\Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta}\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}\Big)\|u_s^\sharp\|_{\mathcal{H}^2}\end{aligned}$$

and the same reasoning gives

$$\|\Psi^s(u) - \Psi_\varepsilon^s(u_\varepsilon)\|_{L^2} \lesssim_{s,\Xi} \|u - u_\varepsilon\|_{L^2} + \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Thus one completes the proof with the bound $\|\Gamma - \Gamma_\varepsilon\|_{\mathcal{H}^\beta\to\mathcal{H}^\beta}$ from Lemma 4.5.

$\square$

The symmetry of H immediately follows.

Corollary. *The operator H is symmetric.*

Proof : Let $u, v \in \mathcal{D}_\Xi$ and consider $u^\sharp := \Phi^s(u)$ and $v^\sharp := \Phi^s(v)$ for $s < s_0(\Xi)$. Since H_ε is a symmetric operator, we have

$$\langle Hu, v\rangle = \lim_{\varepsilon\to 0}\langle H_\varepsilon\Gamma_\varepsilon u^\sharp, \Gamma_\varepsilon v^\sharp\rangle = \lim_{\varepsilon\to 0}\langle \Gamma_\varepsilon u^\sharp, H_\varepsilon\Gamma_\varepsilon v^\sharp\rangle = \langle u, Hv\rangle$$

using that $H_\varepsilon\Gamma_\varepsilon$ converges to $H\Gamma$ and Γ_ε to Γ in norm convergence.

□

4.3.2 – The operator is self-adjoint

The next Proposition states that the quadratic form associated to H is bounded from below by the $\mathcal{H}^1$ norm of $u^\sharp$. This weak coercivity property will give below self-adjointness with the Babuška-Lax-Milgram Theorem. This was already used in the work [38] of Gubinelli, Ugurcan and Zachhuber, where the proof of self-adjointness relies on the reasoning of almost duality encoded in the operator A. For $s \in (0,1)$ and $\delta > 0$, introduce the constant

$$\begin{aligned} m_\delta^1(\Xi, s) := k\Big\{& x(1+x^2) + s^{\frac{\alpha-\beta}{4}}x^2(1+x^3) + s^{\frac{\alpha-2}{2}}x(1+x^2) + s^{\frac{\alpha-4}{2}}x \\ &+ \delta^{-\frac{\beta}{1-\beta}}\Big(x(1+x^2) + s^{\frac{\alpha-\beta}{4}}x^2(1+x)\Big)^{\frac{\beta}{1-\beta}}\Big(1 + s^{\frac{\alpha}{4}}x(1+x)\Big)\Big\} \end{aligned}$$

where $\beta = \frac{1}{2}(\frac{2}{3}+\alpha)$ and for a constant $k > 0$ large enough depending only on M and L while the index "1" refers to $\mathcal{H}^1$. In particular, it depends polynomialy on the enhanced noise and diverges as s or δ goes to 0.

Proposition 4.9. *Let $u \in \mathcal{D}_\Xi$ and $s > 0$. For any $\delta > 0$, we have*

$$(1-\delta)\langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle \le \langle u, Hu\rangle + m_\delta^1(\Xi, s)\|u\|_{L^2}^2$$

and

$$(1-\delta)\langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle \le \langle u, H_\varepsilon u\rangle + m_\delta^1(\Xi, s)\|u\|_{L^2}^2$$

where $u_s^\sharp = \Phi^s(u)$.

Proof : For $u \in \mathcal{D}_\Xi$, we have

$$Hu = Lu_s^\sharp + \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R(u) + \Psi^s(u)$$

with $u_s^\sharp = \Phi^s(u) \in \mathcal{H}^2$. Thus

$$\begin{aligned} \langle u, Lu_s^\sharp\rangle &= \big\langle \widetilde{\mathsf{P}}_u^s X_1, Lu_s^\sharp\big\rangle + \big\langle \widetilde{\mathsf{P}}_u^s X_2, Lu_s^\sharp\big\rangle + \big\langle u_s^\sharp, Lu_s^\sharp\big\rangle \\ &= \big\langle \mathsf{P}_u^s LX_1, u_s^\sharp\big\rangle + \big\langle \mathsf{P}_u^s LX_2, u_s^\sharp\big\rangle + \big\langle \nabla u_s^\sharp, \nabla u_s^\sharp\big\rangle \end{aligned}$$

and this yields

$$\begin{aligned} \langle u, Hu\rangle &= -\big\langle \mathsf{P}_u^s \xi, u_s^\sharp\big\rangle + \big\langle \mathsf{P}_u^s LX_2, u_s^\sharp\big\rangle + \big\langle \nabla u_s^\sharp, \nabla u_s^\sharp\big\rangle \\ &\quad + \big\langle u, \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi)\big\rangle + \big\langle u, R(u) + \Psi^s(u)\big\rangle \\ &= -\mathsf{A}(u, \xi, u_s^\sharp) + \big\langle \mathsf{P}_u^s LX_2, u_s^\sharp\big\rangle + \big\langle \nabla u_s^\sharp, \nabla u_s^\sharp\big\rangle + \big\langle u, \mathsf{P}_\xi u_s^\sharp\big\rangle \\ &\quad + \big\langle u, R(u) + \Psi^s(u)\big\rangle + \big\langle (\mathsf{P}_u - \mathsf{P}_u^s)\xi, u_s^\sharp\big\rangle \end{aligned}$$

where $\mathsf{A}(u,\xi,u^\sharp)=\big\langle \mathsf{P}_u\xi,u^\sharp\big\rangle-\big\langle u,\Pi(u^\sharp,\xi)\big\rangle$. For $\beta:=\frac{1}{2}(\frac{2}{3}+\alpha)$, we have

$$\begin{aligned}\big|\big\langle u,R(u)\big\rangle\big| &\lesssim \|u\|_{L^2}\|R(u)\|_{L^2}\lesssim x(1+x^2)\|u\|_{L^2}\|u\|_{\mathcal{H}^\beta},\\ \big|\big\langle u,\mathsf{P}_\xi u_s^\sharp\big\rangle\big| &\lesssim \|u\|_{\mathcal{H}^\beta}\|\mathsf{P}_\xi u_s^\sharp\|_{\mathcal{C}^{2\beta-2}}\lesssim x\|u\|_{\mathcal{H}^\beta}\|u_s^\sharp\|_{\mathcal{H}^\beta},\\ \big|\big\langle \mathsf{P}_u LX_2,u_s^\sharp\big\rangle\big| &\lesssim \|\mathsf{P}_u LX_2\|_{\mathcal{H}^{2\beta-2}}\|u_s^\sharp\|_{\mathcal{H}^\beta}\lesssim x^2\|u\|_{L^2}\|u_s^\sharp\|_{\mathcal{H}^\beta}.\end{aligned}$$

Using Proposition 1.5, we have

$$\big|\mathsf{A}(u,\xi,u_s^\sharp)\big|\lesssim \|\xi\|_{\mathcal{C}^{\alpha-2}}\|u\|_{\mathcal{H}^\beta}\|u_s^\sharp\|_{\mathcal{H}^\beta}\lesssim x\|u\|_{\mathcal{H}^\beta}\|u_s^\sharp\|_{\mathcal{H}^\beta}.$$

Finally, we have

$$\begin{aligned}\big|\big\langle u,\Psi^s(u)\big\rangle\big| &\lesssim \|u\|_{L^2}\|\Psi^s(u)\|_{L^2}\lesssim s^{\frac{\alpha-2}{2}}x(1+x^2)\|u\|_{L^2}^2\\ \big|\big\langle (\mathsf{P}_u-\mathsf{P}_u^s)\xi,u_s^\sharp\big\rangle\big| &\lesssim \|(\mathsf{P}_u-\mathsf{P}_u^s)\xi\|_{L^2}\|u_s^\sharp\|_{L^2}\lesssim s^{\frac{\alpha-4}{2}}x\|u\|_{L^2}\|u_s^\sharp\|_{L^2}\end{aligned}$$

with Proposition A.7 in Appendix. Since $u\in\mathcal{D}_\Xi$, we have

$$\|u\|_{\mathcal{H}^\beta}\leq\|u_s^\sharp\|_{\mathcal{H}^\beta}+\frac{m}{\alpha-\beta}s^{\frac{\alpha-\beta}{4}}x(1+x)\|u\|_{L^2}$$

hence there exists $k>0$ such that

$$\begin{aligned}\big\langle\nabla u_s^\sharp,\nabla u_s^\sharp\big\rangle\leq&\big\langle u,Hu\big\rangle+k\Big(x(1+x^2)+s^{\frac{\alpha-\beta}{4}}x^2(1+x^3)+s^{\frac{\alpha-2}{2}}x(1+x^2)+s^{\frac{\alpha-4}{2}}x\Big)\|u\|_{L^2}^2\\ &+k\Big(x(1+x^2)+s^{\frac{\alpha-\beta}{4}}x^2(1+x)\Big)\|u_s^\sharp\|_{\mathcal{H}^\beta}.\end{aligned}$$

Since $0<\beta<1$, we have for any $t>0$

$$\begin{aligned}\|u_s^\sharp\|_{\mathcal{H}^\beta}^2&\lesssim\bigg\|\int_0^t(t'L)e^{-t'L}u_s^\sharp\frac{\mathrm{d}t'}{t'}\bigg\|_{\mathcal{H}^\beta}^2+\big\|e^{-tL}u_s^\sharp\big\|_{\mathcal{H}^\beta}^2\\ &\lesssim t^{1-\beta}\|u_s^\sharp\|_{\mathcal{H}^1}^2+t^{-\beta}\Big(1+s^{\frac{\alpha}{4}}x(1+x)\Big)^2\|u\|_{L^2}^2.\end{aligned}$$

Given any $\delta>0$, we set

$$t=\left(\frac{\delta}{k'\Big(x(1+x^2)+s^{\frac{\alpha-\beta}{4}}x^2(1+x)\Big)}\right)^{\frac{1}{1-\beta}}$$

where $k'>0$ the constant from the previous inequality and this yields

$$(1-\delta)\big\langle\nabla u_s^\sharp,\nabla u_s^\sharp\big\rangle\leq\big\langle u,Hu\big\rangle+m_\delta^1(\Xi,s)\|u\|_{L^2}.$$

The same computations show

$$(1-\delta)\langle\nabla u_s^\sharp,\nabla u_s^\sharp\rangle\leq\langle u,H_\varepsilon u\rangle+m_\delta^1(\Xi_\varepsilon,s)\|u\|_{L^2}^2.$$

Since $\|\Xi_\varepsilon-\Xi\|_\alpha$ goes to 0 as ε goes to 0, the result holds uniformly in ε with $m_\delta^1(\Xi,s)$.

□

This implies that H is almost surely bounded below by the random variable $-m_\delta^1(\Xi,s)$ for any $\delta>0$ and $s>0$. Using the Babuška-Lax-Milgram Theorem, one gets an invertible operator via the solution of

$$(H+k_\Xi)u=v$$

for $k_\Xi>m_\delta^1(\Xi,s)$ and $v\in L^2$.

Proposition 4.10. *Let $\delta \in (0,1)$ and $s > 0$. Then for any constant $k_\Xi > m_\delta^1(\Xi, s)$, the operators $H + k_\Xi$ and $H_\varepsilon + k_\Xi$ are invertibles. Moreover the operators*

$$\big(H + k_\Xi\big)^{-1} : L^2 \to \mathcal{D}_\Xi$$
$$\big(H_\varepsilon + k_\Xi\big)^{-1} : L^2 \to \mathcal{H}^2$$

are bounded.

Proof : We want to use the Theorem of Babuška-Lax-Milgram, see [5]. This is a generalization of the Lax-Milgram Theorem with a weaker condition of coercivity. Since $k_\Xi > m_\delta^1(\Xi, s)$, Proposition 4.9 gives

$$\big(k_\Xi - m_\delta^1(\Xi, s)\big)\|u\|_{L^2}^2 < \big\langle (H + k_\Xi)u, u\big\rangle$$

for $u \in \mathcal{D}_\Xi$. Considering the norm

$$\|u\|_{\mathcal{D}_\Xi}^2 = \|u\|_{L^2}^2 + \|u_s^\sharp\|_{\mathcal{H}^2}^2$$

on $\mathcal{D}_\Xi$, this yields a weakly coercive operator using Proposition 4.6 in the sense that

$$\|u\|_{\mathcal{D}_\Xi} \lesssim_\Xi \|(H + k_\Xi)u\|_{L^2} = \sup_{\|v\|_{L^2}=1} \big\langle (H + k_\Xi)u, v\big\rangle$$

for any $u \in \mathcal{D}_\Xi$. Moreover, the bilinear map

$$\begin{array}{rccc} B: & \mathcal{D}_\Xi \times L^2 & \to & \mathbb{R} \\ & (u,v) & \mapsto & \big\langle (H + k_\Xi)u, v\big\rangle \end{array}$$

is continuous since Proposition 4.6 implies

$$|B(u,v)| \le \|(H + k_\Xi)u\|_{L^2}\|v\|_{L^2} \lesssim_\Xi \|u\|_{\mathcal{D}_\Xi}\|v\|_{L^2}$$

for $u \in \mathcal{D}_\Xi$ and $v \in L^2$. The last condition we need is that for any $v \in L^2\backslash\{0\}$, we have

$$\sup_{\|u\|_{\mathcal{D}_\Xi}=1} |B(u,v)| > 0.$$

Let assume that there exists $v \in L^2$ such that $B(u,v) = 0$ for all $u \in \mathcal{D}_\Xi$. Then

$$\forall u \in \mathcal{D}_\Xi, \quad \langle u, v\rangle_{\mathcal{D}_\Xi, \mathcal{D}_\Xi^*} = 0.$$

hence $v = 0$ as an element of $\mathcal{D}_\Xi^*$. By density of $\mathcal{D}_\Xi$ in L^2, this implies $v = 0$ in L^2 hence the property we want. By the Theorem of Babuška-Lax-Milgram, for any $f \in L^2$ there exists a unique $u \in \mathcal{D}_\Xi$ such that

$$\forall v \in L^2, \quad B(u,v) = \langle f, v\rangle.$$

Moreover, we have $\|u\|_{\mathcal{D}_\Xi} \lesssim_\Xi \|f\|_{L^2}$ hence the result for $(H + k_\Xi)^{-1}$. The same argument works for $H_\varepsilon + k_\Xi$ since Proposition 4.9 also holds for H_ε with bounds uniform in ε.

$\square$

Using that a closed symmetric operator on a Hilbert space is self-adjoint if it has at least one real value in its resolvent set, this immediatly implies that H and H_ε are self-adjoint, see [56]. Moreover, the resolvant is a compact operator from L^2 to itself since $\mathcal{D}_\Xi \subset \mathcal{H}^\beta$ for any $\beta \in [0, \alpha)$ hence the following result.

Corollary 4.11. *The operators H and H_ε are self-adjoint with pure point spectrum $\big(\lambda_n(\Xi)\big)_{n\geq 1}$ and $\big(\lambda_n(\Xi_\varepsilon)\big)_{n\geq 1}$ which are nondecreasing diverging sequences without accumulation points. Moreover, we have*

$$L^2 = \bigoplus_{n\geq 1} \mathrm{Ker}\big(H - \lambda_n(\Xi)\big)$$

with each kernel being of finite dimension. We finally have the min-max principle

$$\lambda_n(\Xi) = \inf_D \sup_{u\in D; \|u\|_{L^2}=1} \langle Hu, u\rangle$$

where D is any n-dimensional subspace of $\mathcal{D}_\Xi$ that can also be given as

$$\lambda_n(\Xi) = \sup_{v_1,\dots,v_{n-1}\in L^2} \inf_{\substack{u\in \mathrm{Vect}(v_1,\dots,v_{n-1})^\perp \\ \|u\|_{L^2}=1}} \langle Hu, u\rangle.$$

4.3.3 – Comparison between the spectrum of *H* and *L*

A natural question now is to estimate the size of the eigenvalues of H and try to get back geometric informations on the manifold M as one can do from the Laplacian. Let λ be an eigenvalue of H and $u \in \mathcal{D}_\Xi$ such that

$$Hu = \lambda u.$$

Then there exists $u^\sharp \in \mathcal{H}^2$ such that $u = \Gamma u^\sharp$ thus

$$H\Gamma u^\sharp = \lambda \Gamma u^\sharp.$$

This yields

$$H\Gamma u^\sharp = \lambda u^\sharp + \lambda\big(\Gamma - \mathrm{Id}\big)u^\sharp$$

hence one can relate the spectrum of H to the one of $H\Gamma$ and the parameter s measures the error since

$$\big(\Gamma - \mathrm{Id}\big)u^\sharp = \widetilde{\mathsf{P}}^s_{\Gamma u^\sharp} X_1 + \widetilde{\mathsf{P}}^s_{\Gamma u^\sharp} X_2.$$

And since $H\Gamma$ is a perturbation of L, one can relate the spectrum of $H\Gamma$ to the spectrum of L, as stated in the following Proposition using the min-max result. We denote by $(\lambda_n)_{n\geq 1}$ the non-decreasing positive sequence of the eigenvalues of L, since it corresponds to the case $\Xi = 0$. For $s \in (0,1)$ and $\delta > 0$, introduce the constant

$$m_\delta^+(\Xi, s) := (1+\delta)\Big(1 + \frac{m}{\alpha} s^{\frac{\alpha}{4}} x(1+x)\Big).$$

If $s < s_0(\Xi)$, we also introduce

$$m_\delta^-(\Xi, s) := (1-\delta)\frac{1}{1 - \frac{m}{\alpha} s^{\frac{\alpha}{4}} x(1+x)}.$$

In particular, the constants depend polynomialy on the enhanced noise Ξ and converge to 1 as δ and s goes to 0. Moreover, $m_\delta^-(\Xi, s)$ diverges as s goes to $s_0(\Xi)$. Write $a, b \leq c$ to mean that we have both $a \leq c$ and $b \leq c$.

Proposition 4.12. *Let $s \in (0,1)$ and $\delta > 0$. Given any $n \in \mathbb{Z}^+$, we have*

$$\lambda_n(\Xi), \lambda_n(\Xi_\varepsilon) \leq m_\delta^+(\Xi, s)\lambda_n + (1+\delta)\big(1 + \frac{m}{\alpha} s^{\frac{\alpha}{4}} x(1+x)\big) + m_\delta^2(\Xi, s).$$

If moreover $s < s_0(\Xi)$, we have

$$\lambda_n(\Xi), \lambda_n(\Xi_\varepsilon) \geq m_\delta^-(\Xi, s)\lambda_n - m_\delta^1(\Xi, s).$$

Proof : Let $u_1^\sharp,\dots,u_n^\sharp\in\mathcal{H}^2$ be an orthonormal family of eigenfunctions of L associated to $\lambda_1,\dots,\lambda_n$ and consider

$$u_i := \Gamma u_i^\sharp \in \mathcal{D}_\Xi$$

for $1\le i\le n$. Since Γ is invertible, the family $(u_1,\dots,u_n)$ is free thus the min-max representation of $\lambda_n(\Xi)$ yields

$$\lambda_n(\Xi) \le \sup_{\substack{u\in \mathrm{Vect}(u_1,\dots,u_n)\\ \|u\|_{L^2}=1}} \langle Hu,u\rangle.$$

Given any normalised $u\in\mathrm{Vect}(u_1,\dots,u_n)$, we have

$$\langle Hu,u\rangle \le \|Hu\|_{L^2} \le (1+\delta)\|u_s^\sharp\|_{\mathcal{H}^2} + m_\delta^2(\Xi,s)$$

for $u_s^\sharp=\Phi^s(u)$ using Proposition 4.6. Moreover

$$\|u_s^\sharp\|_{\mathcal{H}^2} \le (1+\lambda_n)\|u_s^\sharp\|_{L^2} \le (1+\lambda_n)\Big(1+\frac{m}{\alpha}s^{\frac{\alpha}{4}}x(1+x)\Big)$$

hence the upper bound

$$\lambda_n(\Xi) \le m_\delta^+(\Xi,s)\lambda_n + 1 + \frac{m}{\alpha}s^{\frac{\alpha}{4}}x(1+x) + m_\delta^2(\Xi,s).$$

For the lower bound, we use the min-max representation of $\lambda_n(\Xi)$ under the form

$$\lambda_n(\Xi) = \sup_{v_1,\dots,v_{n-1}\in L^2}\ \inf_{\substack{u\in\mathrm{Vect}(v_1,\dots,v_{n-1})^\perp\\ \|u\|_{L^2}=1}} \langle Hu,u\rangle.$$

Introducing

$$F := \mathrm{Vect}(u_m; m\ge n),$$

we have that $F^\perp$ is a subspace of L^2 of finite dimension $n-1$ thus there exists a orthogonal family $(v_1,\dots,v_{n-1})$ such that $F^\perp=\mathrm{Vect}(v_1,\dots,v_{n-1})$. Since F is a closed subspace of L^2 as an intersection of hyperplans, we have $F=\mathrm{Vect}(v_1,\dots,v_{n-1})^\perp$ hence

$$\lambda_n(\Xi) \ge \inf_{\substack{u\in F\\ \|u\|_{L^2}=1}} \langle Hu,u\rangle.$$

Let $u\in F$ with $\|u\|_{L^2}=1$. Using Proposition 4.9, we have

$$\begin{aligned}\langle Hu,u\rangle &\ge (1-\delta)\langle\nabla u_s^\sharp,\nabla u_s^\sharp\rangle - m_\delta^1(\Xi,s)\\ &\ge (1-\delta)\langle u_s^\sharp, Lu_s^\sharp\rangle - m_\delta^1(\Xi,s)\\ &\ge (1-\delta)\lambda_n\|u_s^\sharp\|_{L^2}^2 - m_\delta^1(\Xi,s).\end{aligned}$$

Finally using Proposition 4.4 for $s<s_0(\Xi)$, we get

$$\langle Hu,u\rangle \ge \frac{1-\delta}{1-\frac{m}{\alpha}s^{\frac{\alpha}{4}}x(1+x)}\lambda_n - m_\delta^1(\Xi,s)$$

and the proof is complete.

$\square$

There is a wide range of choices for the constants $s \in (0,1)$ and $\delta > 0$. For example, one can take

$$s = \left(\frac{\alpha\delta}{mx(1+x)}\right)^{\frac{4}{\alpha}}$$

for any $\delta \in (0,1)$ and get

$$\lambda_n - m_\delta^1 \leq \lambda_n(\Xi) \leq (1+\delta)^2\lambda_n + 1 + \delta + m_\delta^2$$

for explicit constants m_δ^1 and m_δ^2, where the lower bound holds since $\delta < 1$ gives $s < s_0(\Xi)$. This implies the following estimate for the tail of all the eigenvalues. A more precise result of this type was already obtained in [45] by Labbé in the flat case for λ to $-\infty$ with $a = 1$ where he also obtained a lower bound on the convergence of the form

$$e^{-a_n\lambda} \leq \mathbb{P}(\lambda_n(\Xi) \leq -\lambda) \leq e^{-b_n\lambda}$$

for $\lambda > 0$ large enough and $a_n > b_n > 0$ two constants. Here we get upper bounds for λ to $+\infty$ and $-\infty$.

Corollary 4.13. *For any $n \in \mathbb{Z}^+$ and $\lambda \in \mathbb{R}$, we have*

$$1 - me^{-h(\lambda - 2\lambda_n)^{\frac{1}{12}}} \leq \mathbb{P}\big(\lambda_n(\Xi) \leq \lambda\big) \leq me^{-h(\lambda_n - \lambda)^{\frac{1}{5}}}$$

where $m = \mathbb{E}\big[e^{h\|\Xi\|_{\mathcal{X}^\alpha}}\big]$.

Proof: Fix $\delta \in (0,1)$ and let $\lambda \in \mathbb{R}$. Denote $m_1 = m_\delta^1$ and $m_2 = m_\delta^2$. We have

$$\mathbb{P}\big(\lambda_n(\Xi) \leq \lambda\big) \leq \mathbb{P}\big(\lambda_n - m_1 \leq \lambda\big)$$

and

$$\mathbb{P}\big(\lambda_n(\Xi) > \lambda\big) \leq \mathbb{P}\big((1+\delta)\lambda_n + m_2 > \lambda\big)$$

thus

$$\mathbb{P}\big(m_2 \leq \lambda - (1+\delta)\lambda_n\big) \leq \mathbb{P}\big(\lambda_n(\Xi) \leq \lambda\big) \leq \mathbb{P}\big(m_1 \geq -\lambda + \lambda_n\big).$$

There exists two constants $a_1, a_2 > 0$ such that

$$m_i \leq 1 + \|\Xi\|_{\mathcal{X}^\alpha}^{a_i}$$

for $i \in \{1,2\}$, take for example $a_1 = 5$ and $a_2 = 12$. Hence

$$\begin{aligned}
\mathbb{P}\big(m_i \geq y\big) &= \mathbb{P}\big(\|\Xi\|_{\mathcal{X}^\alpha} \geq (y-1)^{\frac{1}{a_i}}\big) \\
&= \mathbb{P}\big(e^{h\|\Xi\|_{\mathcal{X}^\alpha}} \geq e^{hy^{\frac{1}{a_i}}}\big) \\
&\leq e^{-hy^{\frac{1}{a_i}}}\mathbb{E}\big[e^{h\|\Xi\|_{\mathcal{X}^\alpha}}\big]
\end{aligned}$$

using Markov inequality and this yields

$$1 - me^{-h(\lambda - (1+\delta)\lambda_n)^{\frac{1}{a_2}}} \leq \mathbb{P}\big(\lambda_n(\Xi) \leq \lambda\big) \leq me^{-h(\lambda_n - \lambda)^{\frac{1}{a_1}}}$$

where $m = \mathbb{E}\big[e^{h\|\Xi\|_{\mathcal{X}^\alpha}}\big]$.

$\square$

We proved that H_ε converges to H is some sense as ε goes to 0. The following Proposition gives the convergence of $H_\varepsilon + k_\Xi$ to $H + k_\Xi$ in resolvent sense as ε goes to 0. We do not need to explicit the constant, it depends polynomialy on the enhanced noise Ξ.

Proposition 4.14. *Let $s \in (0,1)$ and $\delta > 0$. Then for any constant $k_\Xi > m_\delta^1(\Xi, s)$, we have*

$$\|(H_\varepsilon + k_\Xi)^{-1} - (H + k_\Xi)^{-1}\|_{L^2 \to L^2} \lesssim_{\Xi,s} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, $(H_\varepsilon + k_\Xi)^{-1}$ converges to $(H + k_\Xi)^{-1}$ in norm as operator from L^2 to itself.

Proof: Let $v \in L^2$. Since $H + k : \mathcal{D}_\Xi \to L^2$ is invertible, there exists $u \in \mathcal{D}_\Xi$ such that

$$v = (H + k)u$$

thus

$$\|(H + k)^{-1}v - (H_\varepsilon + k)^{-1}v\|_{L^2} = \|u - (H_\varepsilon + k)^{-1}(H + k)u\|_{L^2}.$$

We introduce $u_\varepsilon := \Gamma_\varepsilon \Phi^s(u)$ which converges to u in L^2 and we have

$$\|u - (H_\varepsilon + k)^{-1}(H + k)u\|_{L^2} \le \|u - u_\varepsilon\|_{L^2} + \|u_\varepsilon - (H_\varepsilon + k)^{-1}(H + k)u\|_{L^2}.$$

Since Lemma 4.5 gives

$$\|u - u_\varepsilon\|_{L^2} \lesssim_{\Xi,s} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha},$$

we only have to bound the second term. We have

$$\begin{aligned}
\|u_\varepsilon - (H_\varepsilon + k)^{-1}(H + k)u\|_{L^2} &= \|(H_\varepsilon + k)^{-1}\big((H_\varepsilon + k)u_\varepsilon - (H + k)u\big)\|_{L^2} \\
&\lesssim \|(H_\varepsilon + k)u_\varepsilon - (H + k)u\|_{L^2} \\
&\lesssim \|H_\varepsilon u_\varepsilon - Hu\|_{L^2} + k\|u_\varepsilon - u\|_{L^2}
\end{aligned}$$

using Proposition 4.8. In the end, we have

$$\|(H + k)^{-1}v - (H_\varepsilon + k)^{-1}v\|_{L^2} \lesssim \|u_s^\sharp\|_{\mathcal{H}^2}\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

hence the result since $(H + k)^{-1} : L^2 \to \mathcal{D}_\Xi$ is continuous.

$\square$

This allows to get a bound on the convergence of $\lambda_n(\Xi_\varepsilon)$ to $\lambda_n(\Xi)$ as ε goes to 0.

Corollary 4.15. *For all $n \in \mathbb{N}^*$, we have*

$$\left|\frac{1}{\lambda_n(\Xi) + k_\Xi} - \frac{1}{\lambda_n(\Xi_\varepsilon) + k_\Xi}\right| \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, this implies

$$|\lambda_n(\Xi) - \lambda_n(\Xi_\varepsilon)| \lesssim_\Xi (\lambda_n(\Xi) + k_\Xi)^2 \ \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

Proof : We use the min-max principle for $(H+k_\Xi)^{-1}$ and $(H_\varepsilon+k_\Xi)^{-1}$ and denote μ_n and $\mu_n^{(\varepsilon)}$ their n-th smallest eigeinvalue with multiplicity. Let $D_n = \mathrm{Vect}(v_1, \ldots, v_n)$ with v_i an eigenfunction associated to $\mu_i^{(\varepsilon)}$ for $1 \leq i \leq n$. Then for all $v \in D_n$ with $\|v\|_{L^2} = 1$, we have

$$\begin{aligned}\big\langle (H+k_\Xi)^{-1}u, u\big\rangle &= \Big\langle \big((H+k_\Xi)^{-1} - (H_\varepsilon + k_\Xi)^{-1}\big)u, u\Big\rangle + \big\langle (H_\varepsilon + k_\Xi)^{-1}u, u\big\rangle \\ &\leq \big\|(H+k_\Xi)^{-1} - (H_\varepsilon + k_\Xi)^{-1}\big\|_{L^2\to L^2} + \mu_n^{(\varepsilon)}\end{aligned}$$

hence with Proposition 5.11 we get

$$\mu_n - \mu_n^{(\varepsilon)} \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Using the same argument with eigeinfunctions associated to $(H+k_\Xi)^{-1}$, we get

$$|\mu_n - \mu_n^{(\varepsilon)}| \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Thus this gives

$$\left|\frac{1}{\lambda_n(\Xi) + k_\Xi} - \frac{1}{\lambda_n(\Xi_\varepsilon) + k_\Xi}\right| \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

and completes the proof with the upper bound on $\lambda_n(\Xi)$.

$\square$

We conclude this section by giving as corollary the Weyl law for the Anderson Hamiltonian H.

Corollary 4.16. *We have*

$$\lim_{\lambda\to\infty} \lambda^{-1}\big|\{n \geq 1; \lambda_n(\Xi) \leq \lambda\}\big| = \frac{\mathrm{Vol}(M)}{4\pi}.$$

Proof : Let $N(\lambda)$ be the number of eigenvalues of the Laplace-Beltrami operator lower than $\lambda \in \mathbb{R}$. Then the lower and upper bounds on the eigenvalues give

$$N\left(\frac{\lambda - m_\delta^2(\Xi)}{1+\delta}\right) \leq \big|\{n \geq 1; \lambda_n(\Xi) \leq \lambda\}\big| \leq N\big(\lambda + m_\delta^1(\Xi)\big)$$

hence the proof is complete using the result for N.

$\square$

Chapter 5

The random magnetic Laplacian

The magnetic Laplacian associated to a magnetic potential A in two dimensions

$$H = (i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2$$

is of interest in the description of a number of physical models. For example, it describes the behavior of a particule in a magnetic field B related to A via

$$B = \nabla \times A = \partial_2 A_1 - \partial_1 A_2.$$

While the case of constant magnetic field has been largely studied, the analysis of the magnetic Laplacian with nonconstant magnetic field gives rise to a number of interesting questions. Motivations to study the magnetic Laplacian are for example the functional formulation of associated PDEs and the analogy with the electric Laplacian $-\Delta + V$ with electric field V. This work is dedicated to the study of the magnetic Laplacian with random singular magnetic field given by the space white noise $B = \xi$. It can be constructed as a distribution with independant random Fourier coefficients with centered normal law of unit variance. In two dimensions, the space white noise belongs almost surely to the Sobolev spaces $\mathcal{H}^{-1-\kappa}$ or Besov-Hölder spaces $\mathcal{C}^{-1-\kappa}$ for any $\kappa > 0$. Since the associated potential A verifies the equation

$$\xi = \partial_2 A_1 - \partial_1 A_2,$$

each component A_1, A_2 are expected to belong to $\mathcal{C}^{-\kappa}$ for any $\kappa > 0$. In particular, its is not even a measurable function and the associated magnetic Laplacian falls in the range of the singular random operator. This is similar to the Anderson Hamiltonian

$$-\Delta + \xi$$

which was defined and studied in the previous Chapter. The random operator introduced and studied in this Chapter is the magnetic analogue of the Anderson Hamiltonian.

The random magnetic Laplacian with white noise magnetic field is formally given by

$$H = L + 2iA \cdot \nabla + A \cdot A + i \cdot \mathrm{div}(A)$$

with $L = -\Delta$ and

$$\xi = \nabla \times A \in \mathcal{C}^{\alpha-2}(\mathbb{T}^2, \mathbb{R})$$

for any $\alpha < 1$. Different choices of potential A can give the same magnetic field B, this is the choice of gauge and our choice is

$$A := \nabla^{\perp}\Phi$$

where $\Phi = \Delta^{-1}\xi \in \mathcal{C}^{\alpha}(\mathbb{T}^2, \mathbb{R})$. It is motivated by the fact that $\mathrm{div}(A) = 0$ hence the operator is simpler to define. We could deal with different choice of gauge, see Section 5.3 for a discussion about this. This leaves us with

$$H = L + 2iA \cdot \nabla + A \cdot A$$

with $A \in \mathcal{C}^{\alpha-1}$ almost surely for any $\alpha < 1$. In particular, the term $A \cdot A$ is singular and one has to give a meaning to it using probabilistic arguments. This is done in Section 5.3 and yields the associated enhanced potential

$$\mathbf{A} = (A, A^2) \in \mathcal{C}^{\alpha-1}(\mathbb{T}^2, \mathbb{R}^2) \times \mathcal{C}^{2\alpha-2}(\mathbb{T}^2, \mathbb{R}).$$

Remark that since A is a distribution of negative Hölder regularity, the singular product $A \cdot A$ is expected to worsen the regularity. Given such an enhanced potential $\mathbf{A}$, we construct in Section 5.1 a dense subspace $\mathcal{D}_{\mathbf{A}} \subset L^2$ such that

$$u \in \mathcal{D}_{\mathbf{A}} \subset L^2 \quad \Longrightarrow \quad Hu \in L^2.$$

In Section 5.2, we show that $(H, \mathcal{D}_{\mathbf{A}})$ is almost surely a self-adjoint operator with pure point spectrum. We also prove that it is the resolvent-limit of

$$H_\varepsilon = L + 2iA_\varepsilon \cdot \nabla + A_\varepsilon^2$$

for any regularisation $\mathbf{A}_\varepsilon = (A_\varepsilon, A_\varepsilon^2) \in C^\infty(\mathbb{T}^2, \mathbb{R}^2) \times C^\infty(\mathbb{T}^2, \mathbb{R})$ such that

$$\lim_{\varepsilon \to 0} \|A - A_\varepsilon\|_{\mathcal{C}^{\alpha-2}} + \|A^2 - A_\varepsilon^2\|_{\mathcal{C}^{2\alpha-2}} = 0.$$

Finally, we construct in Section 5.3 the enhanced potential $\mathbf{A}$ associated to the random magnetic field $B = \xi$. In particular, it is described by the limit of

$$\big(A_\varepsilon, A_\varepsilon^2 - c_\varepsilon\big)$$

as ε goes to 0 where A_ε is a regularisation of A and

$$c_\varepsilon = \mathbb{E}\big[A_\varepsilon(0) \cdot A_\varepsilon(0)\big].$$

In particular, the almost sure singularity of the product $A \cdot A$ implies the need to substract a diverging constant c_ε as ε goes to 0 and a singular random operator has to be interpreted as the description of the limiting behavior of a diverging system. One is interested in the fluctuations of this system in this diverging frame, as the central limit Theorem for a simple random walk. For the case of the Anderson Hamiltonian, see the work [48] of Martin and Perkowski for a nice example.

Our results on the random magnetic Laplacian is the analogue of the ones obtained in Chapter 4 on the Anderson Hamiltonian. This illustrates the flexibility of the paracontrolled calculus approach to singular stochastic PDEs. In particular, this show that the method used here allow to deal with a general class of operators of the form

$$-\Delta + a_1 \cdot \nabla + a_2$$

with rough stochastic scalar fields $a_1, a_2 : M \to \mathbb{R}$ on a Riemaniann manifold M and therefore associated time-dependant PDEs. See Chapter 7 for another example of such operator in the context of the Brox diffusion. A description of the small or large noise limit of the operator would be an interesting question to pursue. For example, the large noise limit is of interest for a description of the parabolic associated equation on the full space. Indeed, the scaling of the white noise implies that the large noise limit correspond to the large volume limit of the torus. For probabilistic motivations to study this limit for the Anderson Hamiltonian, see [24, 44]. From an analyst point of view, this corresponds to the semi-classical limit of

$$-h^2\Delta + \xi$$

and

$$(ih\partial_1 + A_1)^2 + (ih\partial_2 + A_2)^2$$

which is obviously interesting.

In the first Section, we construct the domain and prove density in L^2. We compare the graph norm and the natural norms of the domain which gives the closedness of the operator. We also give an explicit form comparison between the random magnetic Laplacian H and the Laplacian L. In the second Section, we show that the operator is symmetric as a weak limit of the regularised operator. The form comparison of H and L with the Babuška-Lax-Milgram Theorem gives the self-adjointness. Finally, we show that H is the resolvent-limit of the regularised operator H_ε and compare the spectrum of H and L. In particular, this implies an almost sure Weyl-type law for the random magnetic Laplacian. The third Section deals with the construction of the enhanced potential $\mathbf{A}$ built from the noise ξ through a renormalisation procedure. This corresponds to the work [49].

5.1 – Definition of the operator

In this Section, we first construct the domain and show that its natural norms are equivalent to the graph norm of H. In particular, this guarantees the closedness of the operator. Finally, we compare the respective forms associated to H and L.

5.1.1 – Construction of the domain

Fix $\alpha \in (\frac{2}{3}, 1)$ and let $\mathbf{A}$ be an enhanced magnetic potential

$$\mathbf{A} = (A, A^2) \in \mathcal{X}^\alpha := \mathcal{C}^{\alpha-1}(\mathbb{T}^2, \mathbb{R}^2) \times \mathcal{C}^{2\alpha-2}(\mathbb{T}^2, \mathbb{R})$$

with its natural norm

$$\|\mathbf{A}\|_{\mathcal{X}^\alpha} := \|A\|_{\mathcal{C}^{\alpha-1}} + \|A^2\|_{\mathcal{C}^{2\alpha-2}}.$$

For $A \in L^\infty$, the term A^2 can be interpreted as $A \cdot A$ while it is not defined if A is only a distribution. It is enhanced in the sense that one does not have a natural interpretation for $A \cdot A$, this is specified by the additional data of A^2. Section 5.3 is devoted to the particular case of magnetic white noise where A^2 is constructed through a probabilistic renormalisation procedure. Thus we refer as noise-dependent a quantity that depends on this enhanced potential $\mathbf{A}$. For any regular function $u \in C^\infty(\mathbb{T}^2)$, we have

$$2iA \cdot \nabla u + A^2 u = \mathsf{P}_{\nabla u} 2iA + \mathsf{P}_u A^2 + (\sharp)$$

where $(\sharp) \in C^\infty(\mathbb{T}^2)$ with $\mathsf{P}_{\nabla u} 2iA \in \mathcal{H}^{\alpha-1}$ and $\mathsf{P}_u A^2 \in \mathcal{H}^{2\alpha-2}$. Assuming $Hu \in L^2$ yields

$$Lu = Hu - 2iA \cdot \nabla u + A^2 u \in \mathcal{H}^{2\alpha-2}$$

since $2\alpha - 2 < \alpha - 1$ hence u is expected to belong to $\mathcal{H}^{2\alpha}$ by elliptic regularity theory. For $u \in \mathcal{H}^{2\alpha}$, we have

$$\begin{aligned} 2iA \cdot \nabla u + A^2 u &= 2i\mathsf{P}_{\nabla u} A + 2i\mathsf{P}_A \nabla u + 2i\mathsf{\Pi}(\nabla u, A) + \mathsf{P}_u A^2 + \mathsf{P}_{A^2} u + \mathsf{\Pi}(u, A^2) \\ &= (\alpha - 1) + (3\alpha - 2) + (2\alpha - 2) + (4\alpha - 2) \\ &= \mathsf{P}_u A^2 + \mathsf{P}_{\nabla u} 2iA + (3\alpha - 2) \end{aligned}$$

where (β) denotes a term of formal regularity $\mathcal{H}^\beta$ for any $\beta \in \mathbb{R}$. Following the paracontrolled calculus approach, we want to consider a paracontrolled function of the form

$$u = \widetilde{\mathsf{P}}_u X_1 + \widetilde{\mathsf{P}}_{\nabla u} X_2 + u^\sharp$$

with $u^\sharp$ a smoother remainder such that $Hu \in L^2$. Thus we take

$$-LX_1 := A^2 \quad \text{and} \quad -LX_2 := 2iA$$

and define the domain of H as follows.

Definition. *We define the set $\mathcal{D}_\mathbf{A}$ of functions paracontrolled by $\mathbf{A}$ as*

$$\mathcal{D}_\mathbf{A} := \{u \in L^2;\ u - \widetilde{\mathsf{P}}_u X_1 - \widetilde{\mathsf{P}}_{\nabla u} X_2 \in \mathcal{H}^2\}.$$

The domain is defined as

$$\mathcal{D}_\mathbf{A} = \Phi^{-1}(\mathcal{H}^2)$$

with

$$\Phi(u) := u - \widetilde{\mathsf{P}}_u X_1 - \widetilde{\mathsf{P}}_{\nabla u} X_2$$

however the domain could be anything from trivial to dense in L^2. For $s \in (0,1)$, we introduce the map Φ^s as

$$\Phi^s : \left| \begin{array}{ccc} \mathcal{D}_\mathbf{A} & \to & \mathcal{H}^2 \\ u & \mapsto & u - \widetilde{\mathsf{P}}^s_u X_1 - \widetilde{\mathsf{P}}^s_{\nabla u} X_2 \end{array} \right.$$

with $\widetilde{\mathsf{P}}^s$ the paraproduct truncated at scale s; see Section 4.2 in the previous Chapter for the definition and continuity estimates. In particular, the map

$$\Phi^s : \mathcal{H}^\beta \to \mathcal{H}^\beta$$

is a perturbation of the identity for any $\beta \in [0, 2\alpha)$ invertible for s small enough, we denote its inverse Γ. Since $\big(\widetilde{\mathsf{P}}_v - \widetilde{\mathsf{P}}^s_v\big) X$ is a smooth function, the domain is also given by

$$\mathcal{D}_\mathbf{A} = (\Phi^s)^{-1}(\mathcal{H}^2) = \Gamma(\mathcal{H}^2).$$

The reader should keep in mind that Γ implicitely depends on s, we do not keep it in the notation to lighten this work. This parametrisation of the domain will be crucial to prove that the domain is dense in L^2 and to study H. In particular, sharp bounds on the eigenvalues of H are needed to get a Weyl-type law for H. To do so, we need to keep a careful track of the different constants. The reader interested only in the construction of the operator and its self-adjointess can skip these computations. Obtaining sharp bounds requires explicit constants with respect to the parameter s and the regularity exponent in the paracontrolled calculus. For $\beta \in [0, 2\alpha)$, let

$$s_\beta(\mathbf{A}) := \left(\frac{\beta^*}{m\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\right)^{\frac{4}{2\alpha-\beta}}$$

where $\beta^* = 1-\beta$ if $\beta \in [0,1)$ and $\beta^* = 2\alpha-\beta$ if $\beta \in [1, 2\alpha)$ and $m > 0$ is a universal constant. The following Proposition gives regularity estimates for Φ^s and Γ.

Proposition 5.1. *Let $\beta \in [0, 2\alpha)$ and $s \in (0,1)$. We have*

$$\|\Phi^s(u) - u\|_{\mathcal{H}^\beta} \leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{\beta^*}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{\mathcal{H}^\beta}.$$

In particular, $s < s_\beta(\mathbf{A})$ implies that the map Φ^s is invertible and its inverse Γ verifies the bound

$$\|\Gamma u^\sharp\|_{\mathcal{H}^\beta} \leq \frac{1}{1 - m\frac{s^{\frac{2\alpha-\beta}{4}}}{\beta^*}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\|u^\sharp\|_{\mathcal{H}^\beta}.$$

Proof: If $\beta < 1$, the bounds on Φ^s follow directly from

$$\begin{aligned}
\|\widetilde{\mathsf{P}}^s_u X_1 + \widetilde{\mathsf{P}}^s_{\nabla u} X_2\|_{\mathcal{H}^\beta} &\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{1-\beta}\|u\|_{L^2}\|X_1\|_{\mathcal{C}^{2\alpha}} + m\frac{s^{\frac{\alpha+1-\beta}{4}}}{1-\beta}\|\nabla u\|_{\mathcal{H}^{\beta-1}}\|X_2\|_{\mathcal{H}^{\alpha+1}} \\
&\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{1-\beta}\|u\|_{\mathcal{H}^\beta}\big(\|X_1\|_{\mathcal{C}^{2\alpha}} + \|X_2\|_{\mathcal{C}^{\alpha+1}}\big) \\
&\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{1-\beta}\|u\|_{\mathcal{H}^\beta}\|\mathbf{A}\|_{\mathcal{X}^\alpha}.
\end{aligned}$$

For $\beta \in [1, 2\alpha)$, we have

$$\begin{aligned}
\|\widetilde{\mathsf{P}}^s_u X_1 + \widetilde{\mathsf{P}}^s_{\nabla u} X_2\|_{\mathcal{H}^\beta} &\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{2\alpha-\beta}\|u\|_{L^2}\|X_1\|_{\mathcal{C}^\alpha} + m\frac{s^{\frac{\alpha+1-\beta}{2}}}{\alpha+1-\beta}\|\nabla u\|_{L^2}\|X_2\|_{\mathcal{H}^{\alpha+1}} \\
&\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{2\alpha-\beta}\|u\|_{\mathcal{H}^1}\big(\|X_1\|_{\mathcal{C}^{2\alpha}} + \|X_2\|_{\mathcal{C}^{\alpha+1}}\big) \\
&\leq m\frac{s^{\frac{2\alpha-\beta}{4}}}{2\alpha-\beta}\|u\|_{\mathcal{H}^1}\|\mathbf{A}\|_{\mathcal{X}^\alpha}.
\end{aligned}$$

The result for Γ follows since $2\alpha - \beta > 0$.

$\square$

We also consider the associated maps Φ^s_ε and Γ_ε for a regularizion $\mathbf{A}_\varepsilon$ of the enhanced potential. It is defined as

$$\Phi^s_\varepsilon(u) := u - \widetilde{\mathsf{P}}_u X_1^{(\varepsilon)} - \widetilde{\mathsf{P}}_{\nabla u} X_2^{(\varepsilon)}$$

and

$$\Gamma_\varepsilon u^\sharp = \widetilde{\mathsf{P}}_{\Gamma_\varepsilon u^\sharp} X_1^{(\varepsilon)} + \widetilde{\mathsf{P}}_{\nabla\Gamma_\varepsilon u^\sharp} X_2^{(\varepsilon)} + u^\sharp$$

where

$$-LX_1^{(\varepsilon)} := A_\varepsilon^2 \quad \text{and} \quad -LX_2^{(\varepsilon)} := 2iA_\varepsilon.$$

They satisfie the same continuity properties as Φ^s and Γ with bounds uniform with respect to ε. Moreover, we have the following approximation Lemma.

Lemma 5.2. *Let $\beta \in [0, 2\alpha)$ and $s \in (0,1)$. If $s \leq s_\beta(\mathbf{A})$, we have*

$$\|\mathrm{Id} - \Gamma\Gamma_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\beta} \lesssim_{\mathbf{A},s} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, this implies the convergence of Γ_ε to Γ with the bound

$$\|\Gamma - \Gamma_\varepsilon\|_{\mathcal{H}^\beta \to \mathcal{H}^\beta} \lesssim_{\mathbf{A},s} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Proof : Given any $u \in \mathcal{H}^\beta$, we have $u = \Gamma\Gamma^{-1}(u) = \Gamma(u - \widetilde{\mathsf{P}}_u^s X_1 - \widetilde{\mathsf{P}}_{\nabla u}^s X_2)$. We get

$$\begin{aligned}
\|u - \Gamma\Gamma_\varepsilon^{-1}(u)\|_{\mathcal{H}^\beta} &= \left\|\Gamma\big(u - \widetilde{\mathsf{P}}_u^s X_1 - \widetilde{\mathsf{P}}_{\nabla u}^s X_2\big) - \Gamma\big(u - \widetilde{\mathsf{P}}_u^s X_1^{(\varepsilon)} - \widetilde{\mathsf{P}}_{\nabla u}^s X_2^{(\varepsilon)}\big)\right\|_{\mathcal{H}^\beta} \\
&= \left\|\Gamma\Big(\widetilde{\mathsf{P}}_u^s\big(X_1^{(\varepsilon)} - X_1\big) + \widetilde{\mathsf{P}}_{\nabla u}^s\big(X_2^{(\varepsilon)} - X_2\big)\Big)\right\|_{\mathcal{H}^\beta} \\
&\lesssim_{\mathbf{A},s} \left\|\widetilde{\mathsf{P}}_u^s\big(X_1^{(\varepsilon)} - X_1\big) + \widetilde{\mathsf{P}}_{\nabla u}^s\big(X_2^{(\varepsilon)} - X_2\big)\right\|_{\mathcal{H}^\beta} \\
&\lesssim_{\mathbf{A},s} \|\mathbf{A}_\varepsilon - \mathbf{A}\|_{\mathcal{X}^{2\alpha}} \|u\|_{L^2}
\end{aligned}$$

since $s < s_\beta(\mathbf{A})$ implies the continuity of $\Gamma : \mathcal{H}^\beta \to \mathcal{H}^\beta$ and $X_i^{(\varepsilon)} - X_i$ depends linearly on $\mathbf{A}_\varepsilon - \mathbf{A}$ for $i \in \{1, 2\}$. The result on $\Gamma - \Gamma_\varepsilon$ follows from the bound on Γ_ε uniform with respect to ε.

□

This allows to prove density of the domain.

Corollary 5.3. *The domain $\mathcal{D}_\mathbf{A}$ is dense in $\mathcal{H}^\beta$ for every $\beta \in [0, 2\alpha)$.*

Proof : Given $f \in \mathcal{H}^2$, $\Gamma(g_\varepsilon) \in \mathcal{D}_\mathbf{A}$ where $g_\varepsilon = \Gamma_\varepsilon^{-1} f \in \mathcal{H}^2$ thus we can conclude with the previous Lemma that

$$\lim_{\varepsilon \to 0} \|f - \Gamma(g_\varepsilon)\|_{\mathcal{H}^\beta} = 0.$$

The density of $\mathcal{H}^2$ in $\mathcal{H}^\beta$ completes the proof.

□

5.1.2 – First properties of *H*

Since H is formally given by

$$H = L + 2iA \cdot \nabla + A \cdot A$$

with $L = -\Delta$, we are able to define $(H, \mathcal{D}_\mathbf{A})$ as an unbounded operator in L^2 associated to the enhanced potential $\mathbf{A}$.

Definition 5.4. *We define* $H : \mathcal{D}_{\mathbf{A}} \subset L^2 \to L^2$ *as*

$$Hu := Lu^\sharp + R(u)$$

where $u^\sharp = \Phi(u)$ *and*

$$R(u) := \mathsf{P}_{2iA}\nabla u + \Pi(\nabla u, 2iA) + \mathsf{P}_{A^2}u + \Pi(u, A^2) + e^{-L}\big(\mathsf{P}_u A^2 + \mathsf{P}_{\nabla u}2iA\big).$$

The definition of H is independant of the parameter $s \in (0,1)$. As for the Anderson Hamiltonian, it is a very useful tool to get differents bounds on the operator with the different representations

$$Hu = Lu_s^\sharp + R(u) + \Psi^s(u)$$

where $u_s^\sharp := \Phi^s(u)$ and

$$\Psi^s(u) := L\big(\widetilde{\mathsf{P}}_u^s - \widetilde{\mathsf{P}}_u\big)X_1 + L\big(\widetilde{\mathsf{P}}_{\nabla u}^s - \widetilde{\mathsf{P}}_{\nabla u}\big)X_2 \in C^\infty(\mathbb{T}^2).$$

For example, we can compare the graph norm of H

$$\|u\|_H^2 := \|u\|_{L^2}^2 + \|Hu\|_{L^2}^2$$

and the natural norms of the domain

$$\|u\|_{\mathcal{D}_{\mathbf{A}}}^2 := \|u\|_{L^2}^2 + \|\Phi^s(u)\|_{\mathcal{H}^2}^2$$

with the following Proposition provided s is small. Let $\beta := \frac{1}{2}(\frac{4}{3} + 2\alpha)$ and $\delta > 0$. For $s \in (0,1)$ such that $s < s_\beta(\mathbf{A})$, we introduce the constant

$$m_\delta^2(\mathbf{A}, s) := ks^{\frac{\alpha-2}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha} + k\delta^{-\frac{\beta}{2-\beta}}\left(\frac{\|\mathbf{A}\|_{\mathcal{X}^\alpha}}{1 - m s^{\frac{\frac{2\alpha-\beta}{4}}{\beta^*}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\right)^{\frac{2}{2-\beta}}\big(1 + s^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\big)$$

with $k > 0$ a large enough constant depending. In particular, $m_\delta^2(\mathbf{A}, s)$ diverges as s goes to 0 or $s_\beta(\mathbf{A})$ or as δ goes to 0.

Proposition 5.5. *Let* $u \in \mathcal{D}_{\mathbf{A}}$ *and* $s \in (0,1)$ *such that* $s < s_\beta(\mathbf{A})$. *Then for any* $\delta > 0$, *we have*

$$(1-\delta)\|u_s^\sharp\|_{\mathcal{H}^2} \le \|Hu\|_{L^2} + m_\delta^2(\mathbf{A}, s)\|u\|_{L^2}$$

and

$$\|Hu\|_{L^2} \le (1+\delta)\|u_s^\sharp\|_{\mathcal{H}^2} + m_\delta^2(\mathbf{A}, s)\|u\|_{L^2}$$

with $u_s^\sharp = \Phi^s(u)$.

Proof: Recall that for any $s \in (0,1)$, the operator is given by

$$Hu = Lu_s^\sharp + R(u) + \Psi^s(u)$$

thus we need to bound R and Ψ^s. For $u \in \mathcal{D}_{\mathbf{A}}$, we have

$$\begin{aligned}\|\mathsf{P}_{2iA}\nabla u + \Pi(\nabla u, 2iA)\|_{L^2} &\lesssim \|2iA\|_{\mathcal{C}^{\alpha-1}}\|u\|_{\mathcal{H}^\beta}\\ \|\mathsf{P}_{A^2}u + \Pi(u, A^2)\|_{L^2} &\lesssim \|A^2\|_{\mathcal{C}^{2\alpha-2}}\|u\|_{\mathcal{H}^\beta}\end{aligned}$$

hence

$$\|R(u)\|_{L^2} \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{\mathcal{H}^\beta}.$$

We also have

$$\|\Psi^s(u)\|_{L^2} \lesssim \|(\widetilde{\mathsf{P}}_u - \widetilde{\mathsf{P}}^s_u)X_1 + (\widetilde{\mathsf{P}}_{\nabla u} - \widetilde{\mathsf{P}}^s_{\nabla u})X_2\|_{\mathcal{H}^2} \lesssim s^{\frac{\alpha-2}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{L^2}.$$

For $s < s_\beta(\mathbf{A})$, Proposition 5.1 gives

$$\|u\|_{\mathcal{H}^\beta} \leq \frac{1}{1 - m\frac{s^{\frac{2\alpha-\beta}{4}}}{\beta^*}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\|u_s^\sharp\|_{\mathcal{H}^\beta}$$

thus we get

$$\|Hu - Lu_s^\sharp\|_{L^2} \lesssim \frac{\|\mathbf{A}\|_{\mathcal{X}^\alpha}}{1 - m\frac{s^{\frac{2\alpha-\beta}{4}}}{\beta^*}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\|u_s^\sharp\|_{\mathcal{H}^\beta} + s^{\frac{\alpha-2}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{L^2}.$$

Since $0 < \beta < 2$, we have for any $t > 0$

$$\begin{aligned}
\|u_s^\sharp\|_{\mathcal{H}^\beta} &\lesssim \left\|\int_0^t (t'L)e^{-t'L}u_s^\sharp\frac{\mathrm{d}t'}{t'}\right\|_{\mathcal{H}^\beta} + \left\|e^{-tL}u_s^\sharp\right\|_{\mathcal{H}^\beta} \\
&\lesssim t^{\frac{2-\beta}{2}}\|u_s^\sharp\|_{\mathcal{H}^2} + t^{-\frac{\beta}{2}}\|u_s^\sharp\|_{L^2} \\
&\lesssim t^{\frac{2-\beta}{2}}\|u_s^\sharp\|_{\mathcal{H}^2} + t^{-\frac{\beta}{2}}\Big(1 + s^{\frac{2\alpha}{4}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\Big)\|u\|_{L^2}.
\end{aligned}$$

For any $\delta > 0$, take

$$t = \left(\frac{\delta\big(1 - m\frac{s^{\frac{2\alpha-\beta}{4}}}{\beta^*}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\big)}{k\|A\|_{\mathcal{X}^\alpha}}\right)^{\frac{2}{2-\beta}}$$

with k the constant from the previous inequality. This yields

$$\|Lu_s^\sharp - Hu\|_{L^2} \lesssim m_\delta^2(\mathbf{A}, s)\|u\|_{L^2} + \delta\|u_s^\sharp\|_{\mathcal{H}^2}.$$

and completes the proof.

□

Remark : *In comparison with the Anderson Hamiltonian from the Chapter 4*

$$u \mapsto -\Delta u + u\xi$$

where the space white noise can be interpreted as an electric potential, one needs s small for these bounds to hold here. In fact, one could perform the same kind of expansion with

$$LX_2 = \mathsf{P}_{\nabla X_1}2iA$$

at the price of a nonlinear dependance of X_2 with respect to $\mathbf{A}$ in order to bypass the smallness condition on s. This would change the different bounds one get for Φ^s and Γ but still yield a self-adjoint operator that is the limit of the regularised H_ε. Theorem XIII.26 *from [56] guarantees that the different choice of construction coincide provided that H is self-adjoint.*

In particular, this implies that $(H, \mathcal{D}_{\mathbf{A}})$ is a closed operator in L^2.

Proposition 5.6. *The operator H is closed on its domain $\mathcal{D}_{\mathbf{A}}$.*

Proof : Let $(u_n)_{n\geq 0} \subset \mathcal{D}_{\mathbf{A}}$ be a sequence such that

$$u_n \to u \quad \text{in } L^2 \quad \text{and} \quad Hu_n \to v \quad \text{in } L^2.$$

Proposition 5.5 gives that $\big(\Phi^s(u_n)\big)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}^2$ hence converges to $u_s^\sharp \in \mathcal{H}^2$ for $s < s_\beta(\mathbf{A})$. Since $\Phi^s : L^2 \to L^2$ is continuous, we have $\Phi^s(u) = u_s^\sharp$ hence $u \in \mathcal{D}_{\mathbf{A}}$. Finally, we have

$$\begin{aligned}\|Hu - v\|_{L^2} &\leq \|Hu - Hu_n\|_{L^2} + \|Hu_n - v\|_{L^2}\\ &\lesssim_{\mathbf{A}} \|u_s^\sharp - \Phi^s(u_n)\|_{\mathcal{H}^2} + \|u - u_n\|_{L^2} + \|Hu_n - v\|_{L^2}\end{aligned}$$

hence $Hu = v$ and H is closed on $\mathcal{D}_{\mathbf{A}}$.

□

We conclud this Section by computing the Hölder regularity of the functions in the domain. Remark that there is no gain with respect to the Anderson Hamiltonian since one is limited by the embedding of $\mathcal{H}^2$ in two dimension of the remainder.

Corollary. *We have*

$$\mathcal{D}_{\mathbf{A}} \subset \mathcal{C}^{1-\kappa}$$

for any $\kappa > 0$.

Proof : The Besov embedding in two dimensions implies

$$\mathcal{H}^2 \hookrightarrow \mathcal{B}^1_{\infty,2} \hookrightarrow L^\infty$$

and $\Phi^s : L^\infty \to L^\infty$ is also invertible for s small enough hence

$$\mathcal{D}_{\mathbf{A}} = (\Phi^s)^{-1}(\mathcal{H}^2) \subset L^\infty.$$

First for $u \in \mathcal{D}_{\mathbf{A}}$, we have

$$\begin{aligned}\|u\|_{\mathcal{C}^\alpha} &\lesssim \|u\|_{L^\infty}\|X_1\|_{\mathcal{C}^\alpha} + \|\nabla u\|_{\mathcal{C}^{-1}}\|X_2\|_{\mathcal{C}^{\alpha+1}} + \|u^\sharp\|_{\mathcal{C}^\alpha}\\ &\lesssim_{\mathbf{A}} \|u\|_{L^\infty} + \|u^\sharp\|_{\mathcal{H}^2}.\end{aligned}$$

Finally, this gives

$$\begin{aligned}\|u\|_{\mathcal{C}^{1-\kappa}} &\lesssim \|u\|_{L^\infty}\|X_1\|_{\mathcal{C}^{1-\kappa}} + \|\nabla u\|_{\mathcal{C}^{\alpha-1}}\|X_2\|_{\mathcal{C}^{2-\alpha+\kappa}} + \|u^\sharp\|_{\mathcal{C}^{1-\kappa}}\\ &\lesssim_{\mathbf{A}} \|u\|_{L^\infty} + \|u\|_{\mathcal{C}^\alpha} + \|u^\sharp\|_{\mathcal{H}^2}\end{aligned}$$

and the proof is complete. Since $\alpha < 1$ is arbitrary closed to 1, the second computation might appear redundant. The point is that one controls the norm $\mathcal{C}^{1-\kappa}$ for all $\kappa > 0$ for any fixed $\alpha < 1$.

□

5.1.3 – Form comparison between *H* and *L*

We proved in Theorem 5.5 that Hu can be seen as a small perturbation of $Lu^\sharp$ in norm. Here, we prove a similar statement in the quadratic form setting. Let $\eta := \frac{\alpha}{4}$ and $\delta > 0$. For $s \in (0,1)$ such that $s < s_{1-\eta}(\mathbf{A})$, define

$$m_\delta^1(\mathbf{A}, s) := (1 + s^{\frac{\alpha-2}{2}})\|\mathbf{A}\|_{\mathcal{X}^\alpha} + \delta^{-\frac{1-\eta}{\eta}}\left(\frac{\|\mathbf{A}\|_{\mathcal{X}^\alpha}}{(1-\eta^{-1}s^{\frac{2\alpha+\eta-1}{4}}\|\mathbf{A}\|_{\mathcal{X}^\alpha})^2}\right)^{\frac{1}{\eta}}(1 + s^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha})$$

with $k > 0$ a large enough constant depending only on L. In particular, $m_\delta^1(\mathbf{A}, s)$ diverges as s goes to 0 or $s_{1-\eta}(\mathbf{A})$ or as δ goes to 0.

Proposition 5.7. *Let $u \in \mathcal{D}_\mathbf{A}$ and $s \in (0,1)$ such that $s < s_{1-\frac{\alpha}{4}}(\mathbf{A})$. For any $\delta > 0$, we have*

$$(1-\delta)\langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle \le \langle u, Hu\rangle + m_\delta^1(\mathbf{A}, s)\|u\|_{L^2}^2$$

and

$$(1-\delta)\langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle \le \langle u, H_\varepsilon u\rangle + m_\delta^1(\mathbf{A}, s)\|u\|_{L^2}^2$$

where $u_s^\sharp = \Phi^s(u)$.

Proof : For $u \in \mathcal{D}_\mathbf{A}$, recall that

$$Hu = Lu_s^\sharp + R(u) + \Psi^s(u)$$

where $u_s^\sharp = \Phi^s(u) \in \mathcal{H}^2$. We have

$$\begin{aligned}\big\langle u, Lu_s^\sharp\big\rangle &= \big\langle \widetilde{\mathsf{P}}_u^s X_1, Lu_s^\sharp\big\rangle + \big\langle \widetilde{\mathsf{P}}_{\nabla u}^s X_2, Lu_s^\sharp\big\rangle + \big\langle u_s^\sharp, Lu_s^\sharp\big\rangle \\ &= \big\langle \mathsf{P}_u^s LX_1, u_s^\sharp\big\rangle + \big\langle \mathsf{P}_{\nabla u}^s LX_2, u_s^\sharp\big\rangle + \big\langle \nabla u_s^\sharp, \nabla u_s^\sharp\big\rangle\end{aligned}$$

thus

$$\langle u, Hu\rangle = \big\langle \mathsf{P}_u^s LX_1, u_s^\sharp\big\rangle + \big\langle \mathsf{P}_{\nabla u}^s LX_2, u_s^\sharp\big\rangle + \big\langle \nabla u_s^\sharp, \nabla u_s^\sharp\big\rangle + \big\langle u, R(u)\big\rangle + \big\langle u, \Psi^s(u)\big\rangle.$$

For $\eta \le \frac{\alpha}{2}$, we have

$$\begin{aligned}\big\langle \mathsf{P}_u^s LX_1, u_s^\sharp\big\rangle &\lesssim \|\mathsf{P}_u^s LX_1\|_{\mathcal{H}^{2\alpha-2}}\|u_s^\sharp\|_{\mathcal{H}^{1-\eta}} \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{L^2}\|u_s^\sharp\|_{\mathcal{H}^{1-\eta}}, \\ \big\langle \mathsf{P}_{\nabla u}^s LX_2, u_s^\sharp\big\rangle &\lesssim \|\mathsf{P}_{\nabla u}^s LX_2\|_{\mathcal{H}^{\alpha-1-\eta}}\|u_s^\sharp\|_{\mathcal{H}^{1-\eta}} \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{\mathcal{H}^{1-\eta}}\|u_s^\sharp\|_{\mathcal{H}^{1-\eta}}, \\ \big\langle u, \mathsf{P}_{2iA}\nabla u\big\rangle &\lesssim \|u\|_{\mathcal{H}^{1-\eta}}\|\mathsf{P}_{2iA}\nabla u\|_{\mathcal{H}^{\alpha-1-\eta}} \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{\mathcal{H}^{1-\eta}}^2, \\ \big\langle u, \mathsf{P}_{A^2}u + \Pi(u, A^2)\big\rangle &\lesssim \|u\|_{L^2}\|\mathsf{P}_{A^2}u + \Pi(u, A^2)\|_{L^2} \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{L^2}\|u\|_{\mathcal{H}^{1-\eta}}, \\ \big\langle u, \Psi^s(u)\big\rangle &\lesssim \|u\|_{L^2}\|(\mathsf{P}_u - \mathsf{P}_u^s)LX_1 + (\mathsf{P}_{\nabla u} - \mathsf{P}_{\nabla u}^s)LX_2\|_{L^2} \lesssim s^{\frac{\alpha-2}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|_{L^2}^2.\end{aligned}$$

The only term that is not a priori controlled is $\big\langle u, \Pi(2iA, \nabla u)\big\rangle$ since the resonant term is singular if we only suppose that $u \in \mathcal{H}^1$; this is where the almost duality property comes into play. We have

$$\big\langle u, \Pi(2iA, \nabla u)\big\rangle = \big\langle \mathsf{P}_u 2iA, \nabla u\big\rangle + \mathsf{A}(u, 2iA, \nabla u)$$

with the corrector $\mathsf{A}(u, 2iA, \nabla u)$ controlled if $u \in \mathcal{H}^{1-\eta}$ with $\eta < \frac{\alpha}{2}$. The paraproduct is not singular however one can not use better regularity than L^2 for u thus we use an integration by part to get

$$\big\langle \mathsf{P}_u 2iA, \nabla u\big\rangle = -\big\langle \mathrm{div}(\mathsf{P}_u 2iA), u\big\rangle = -\big\langle \mathsf{P}_u \mathrm{div}(2iA), u\big\rangle + \big\langle \mathsf{B}(u, 2iA), u\big\rangle$$

with

$$\mathsf{B}\big(a,(b_1,b_2)\big) := \mathrm{div}\big(\mathsf{P}_a(b_1,b_2)\big) - \mathsf{P}_a\mathrm{div}(b_1,b_2),$$

see Proposition A.10 in Appendix for continuity estimates on B. We have

$$\big\langle u, \Pi(2iA,\nabla u)\big\rangle \lesssim \big|\big\langle \mathsf{B}(u,2iA),u\big\rangle\big| + |\mathsf{A}(u,2iA,\nabla u)| \lesssim \|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|^2_{\mathcal{H}^{1-\eta}}$$

since $\mathrm{div}(A)=0$. Since $s < s_{1-\eta}(\mathbf{A})$, we get

$$\big|\langle u,Hu\rangle - \langle \nabla u_s^\sharp,\nabla u_s^\sharp\rangle\big| \lesssim (1+s^{\frac{\alpha-2}{2}})\|\mathbf{A}\|_{\mathcal{X}^\alpha}\|u\|^2_{L^2} + \frac{\|\mathbf{A}\|_{\mathcal{X}^\alpha}}{\big(1-\eta^{-1}s^{\frac{2\alpha+\eta-1}{4}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\big)^2}\|u_s^\sharp\|^2_{\mathcal{H}^{1-\eta}}.$$

To complete the proof, one only has to interpolate the $\mathcal{H}^{1-\eta}$ norm of $u_s^\sharp$ between its $\mathcal{H}^1$ norm and its L^2 norm which is controlled by the L^2 norm of u, as in the proof of Proposition 5.5. Since $0 < 1-\eta < 1$, we have for any $t>0$

$$\begin{aligned}
\|u_s^\sharp\|_{\mathcal{H}^{1-\eta}} &\lesssim \left\|\int_0^t (t'L)e^{-t'L}u_s^\sharp\frac{\mathrm{d}t'}{t'}\right\|_{\mathcal{H}^{1-\eta}} + \left\|e^{-tL}u_s^\sharp\right\|_{\mathcal{H}^{1-\eta}}\\
&\lesssim t^{\frac{\eta}{2}}\|u_s^\sharp\|_{\mathcal{H}^1} + t^{-\frac{1-\eta}{2}}\|u_s^\sharp\|_{L^2}\\
&\lesssim t^{\frac{\eta}{2}}\|u_s^\sharp\|_{\mathcal{H}^1} + t^{-\frac{1-\eta}{2}}\Big(1+s^{\frac{2\alpha}{4}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\Big)\|u\|_{L^2}.
\end{aligned}$$

For any $\delta>0$, take

$$t = \left(\frac{\delta\big(1-\eta^{-1}s^{\frac{2\alpha+\eta-1}{4}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\big)^2}{k\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\right)^{\frac{2}{\eta}}$$

with k the constant from the previous inequality. This yields

$$\|Lu_s^\sharp - Hu\|_{L^2} \lesssim m_\delta^2(\mathbf{A},s)\|u\|_{L^2} + \delta\|u_s^\sharp\|_{\mathcal{H}^1}.$$

and completes the proof.

□

5.2 – Self-adjointness and spectrum

In this Section, we prove that H is self-adjoint with pure point spectrum. It is symmetric since $H\Gamma$ is the limit in norm of the regularised $H_\varepsilon\Gamma_\varepsilon$ as proved in Section 5.2.1. Hence it is enough to prove that

$$H + k : \mathcal{D}_{\mathbf{A}} \to L^2$$

is surjective for some $k\in\mathbb{R}$, this is the content of Section 5.2.2. In Section 5.2.3, we prove that H_ε converges to H in the stronger resolvent sense. Finally, we give in Section 5.2.4 bounds for the eigenvalues of H using the different representation

$$H = L\Phi^s + R + \Psi^s$$

parametrised by $s\in(0,1)$ from the eigenvalues of L. In particular, it implies a Weyl-type law for H.

5.2.1 – The operator is symmetric

To prove that H is symmetric, we use the regularised operator H_ε. Recall that $(H_\varepsilon, \mathcal{H}^2)$ is self-adjoint and that $\Phi_\varepsilon^s : \mathcal{H}^2 \to \mathcal{H}^2$ is continuous. In some sense, the operator H should be the limit of

$$H_\varepsilon := L + 2iA_\varepsilon \cdot \nabla + A_\varepsilon^2$$

as ε goes to 0 with $\mathbf{A}_\varepsilon := (A_\varepsilon, A_\varepsilon^2)$ a smooth approximation of $\mathbf{A}$ in $\mathcal{X}^\alpha$. Since $\mathcal{D}(H_\varepsilon) = \mathcal{H}^2$, one can not compare directly the operators. However given any $u \in L^2$, we have

$$u = \big(\Gamma \circ \Phi^s\big)(u) = \lim_{\varepsilon \to 0} \big(\Gamma_\varepsilon \circ \Phi^s\big)(u).$$

Thus for $u \in \mathcal{D}_\mathbf{A}$, the approximation $u_\varepsilon := \big(\Gamma_\varepsilon \circ \Phi^s\big)(u)$ belongs to $\mathcal{H}^2$ and one can consider the difference

$$\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} = \|(H\Gamma - H_\varepsilon \Gamma_\varepsilon)u^\sharp\|_{L^2}$$

with $u^\sharp := \Phi^s(u)$. The following Proposition assures the convergence of $H_\varepsilon \Gamma_\varepsilon$ to $H\Gamma$ provided $s < s_\beta(\mathbf{A})$ where $\beta = \frac{1}{2}(\frac{4}{3} + 2\alpha)$.

Proposition 5.8. *Let $u \in \mathcal{D}_\mathbf{A}$ and $s \in (0,1)$ such that $s < s_\beta(\mathbf{A})$. Then*

$$\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} \lesssim_{\mathbf{A},s} \|u_s^\sharp\|_{\mathcal{H}^2} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha}$$

with $u_s^\sharp = \Phi^s(u)$ and $u_\varepsilon := \Gamma_\varepsilon u_s^\sharp$. In particular, this implies that $H_\varepsilon \Gamma_\varepsilon$ converges to $H\Gamma$ in norm as ε goes to 0 as operators from $\mathcal{H}^2$ to L^2.

Proof : We have

$$H_\varepsilon u_\varepsilon = L u_s^\sharp + R_\varepsilon(u_\varepsilon) + \Psi_\varepsilon^s(u_\varepsilon)$$

where R_ε and Ψ_ε^s are defined as R and Ψ^s with $\mathbf{A}_\varepsilon$ instead of $\mathbf{A}$. Since $\frac{4}{3} < \beta < 2\alpha$, we have

$$\begin{aligned}\|R(u) - R_\varepsilon(u_\varepsilon)\|_{L^2} &\le \|R(u - u_\varepsilon)\|_{L^2} + \|(R - R_\varepsilon)(u_\varepsilon)\|_{L^2} \\ &\lesssim_{s,\mathbf{A}} \|u - u_\varepsilon\|_{\mathcal{H}^\beta} + \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha} \|u_\varepsilon\|_{\mathcal{H}^\beta}\end{aligned}$$

and

$$\|\Psi^s(u) - \Psi_\varepsilon^s(u_\varepsilon)\|_{L^2} \lesssim_{s,\mathbf{A}} \|u - u_\varepsilon\|_{L^2} + \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha} \|u\|_{L^2}$$

and the proof is complete since $s < s_\beta(\mathbf{A})$ implies

$$\|u\|_{\mathcal{H}^\beta} \lesssim_{s,\mathbf{A}} \|u_s^\sharp\|_{\mathcal{H}^\beta}.$$

□

The symmetry of H immediately follows.

Corollary. *The operator H is symmetric.*

Proof : Let $u, v \in \mathcal{D}_\mathbf{A}$ and consider $u^\sharp := \Phi^s(u)$ and $v^\sharp := \Phi^s(v)$ for $s < s_\beta(\mathbf{A})$. Since H_ε is a symmetric operator, we have

$$\langle Hu, v\rangle = \lim_{\varepsilon \to 0} \langle H_\varepsilon \Gamma_\varepsilon u_s^\sharp, \Gamma_\varepsilon v_s^\sharp\rangle = \lim_{\varepsilon \to 0} \langle \Gamma_\varepsilon u_s^\sharp, H_\varepsilon \Gamma_\varepsilon v_s^\sharp\rangle = \langle u, Hv\rangle$$

using that $H_\varepsilon \Gamma_\varepsilon$ converges to $H\Gamma$ and Γ_ε to Γ in norm convergence.

□

5.2.2 – The operator is self-adjoint

In this Section we prove that $(H, \mathcal{D}_{\mathbf{A}})$ is self-adjoint. Being closed and symmetric, it is enough to prove that

$$(H+k)u = v$$

admits a solution for some $k \in \mathbb{R}$, see Theorem X.1 in [55]. This is done using the Babuška-Lax-Milgram Theorem and Theorem 5.7 which implies that H is almost surely bounded below for any $\delta \in (0,1)$ and s small enough.

Proposition 5.9. *Let $\delta \in (0,1)$ and $s \in (0,1)$ such that $s < s_{1-\frac{\alpha}{4}}(\mathbf{A})$. For $k > m_\delta^1(\mathbf{A}, s)$, the operators $H+k$ and $H_\varepsilon + k$ are invertibles as unbounded operator in L^2. Moreover the operators*

$$\big(H+k\big)^{-1} : L^2 \to \mathcal{D}_{\mathbf{A}}$$
$$\big(H_\varepsilon+k\big)^{-1} : L^2 \to \mathcal{H}^2$$

are bounded.

Proof : Since $s < s_{1-\frac{\alpha}{4}}(\mathbf{A})$ and $k > m_\delta^1(\mathbf{A}, s)$, Proposition 5.7 gives

$$\big(k - m_\delta^1(\mathbf{A}, s)\big)\|u\|_{L^2}^2 < \big\langle (H+k)u, u\big\rangle$$

for $u \in \mathcal{D}_{\mathbf{A}}$. Considering the norm

$$\|u\|_{\mathcal{D}_{\mathbf{A}}}^2 = \|u\|_{L^2}^2 + \|u_s^\sharp\|_{\mathcal{H}^2}^2$$

on $\mathcal{D}_{\mathbf{A}}$, this yields a weakly coercive operator using Proposition 5.5 in the sense that

$$\|u\|_{\mathcal{D}_{\mathbf{A}}} \lesssim_{\mathbf{A}} \|(H+k)u\|_{L^2} = \sup_{\|v\|_{L^2}=1} \big\langle (H+k)u, v\big\rangle$$

for any $u \in \mathcal{D}_{\mathbf{A}}$. Moreover, the bilinear map

$$\begin{array}{rccc} B: & \mathcal{D}_{\mathbf{A}} \times L^2 & \to & \mathbb{R} \\ & (u,v) & \mapsto & \big\langle (H+k)u, v\big\rangle \end{array}$$

is continuous since Proposition 5.5 implies

$$|B(u,v)| \leq \|(H+k)u\|_{L^2}\|v\|_{L^2} \lesssim_{\mathbf{A}} \|u\|_{\mathcal{D}_{\mathbf{A}}}\|v\|_{L^2}$$

for $u \in \mathcal{D}_{\mathbf{A}}$ and $v \in L^2$. The last condition we need is that for any $v \in L^2 \backslash \{0\}$, we have

$$\sup_{\|u\|_{\mathcal{D}_{\mathbf{A}}}=1} |B(u,v)| > 0.$$

Let assume that there exists $v \in L^2$ such that $B(u,v) = 0$ for all $u \in \mathcal{D}_{\mathbf{A}}$. Then

$$\forall u \in \mathcal{D}_{\mathbf{A}}, \quad \langle u, v\rangle_{\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{A}}^*} = 0.$$

hence $v = 0$ as an element of $\mathcal{D}_{\mathbf{A}}^*$. By density of $\mathcal{D}_{\mathbf{A}}$ in L^2, this implies $v = 0$ in L^2 hence the property we want. By the Theorem of Babuška-Lax-Milgram, for any $f \in L^2$ there exists a unique $u \in \mathcal{D}_{\mathbf{A}}$ such that

$$\forall v \in L^2, \quad B(u,v) = \langle f, v\rangle.$$

Moreover, we have $\|u\|_{\mathcal{D}_{\mathbf{A}}} \lesssim_{\mathbf{A}} \|f\|_{L^2}$ hence the result for $(H+k)^{-1}$. The same argument works for $H_\varepsilon + k$ since Proposition 5.7 also holds for H_ε with bounds uniform in ε.

□

As explain before, this immediatly implies that H is self-adjoint. Moreover, the resolvent is a compact operator from L^2 to itself since $\mathcal{D}_{\mathbf{A}} \subset \mathcal{H}^\beta$ for any $\beta \in [0, 2\alpha)$ hence it has pure point spectrum.

Corollary 5.10. *The operator H is self-adjoint with discret spectrum $\big(\lambda_n(\mathbf{A})\big)_{n\geq 1}$ which is a nondecreasing diverging sequence without accumulation points. Moreover, we have*

$$L^2 = \bigoplus_{n\geq 1} \operatorname{Ker}\big(H - \lambda_n(\mathbf{A})\big)$$

with each kernel being of finite dimension. We finally have the min-max principle

$$\lambda_n(\mathbf{A}) = \inf_D \sup_{u\in D; \|u\|_{L^2}=1} \langle Hu, u\rangle$$

where D is any n-dimensional subspace of $\mathcal{D}_{\mathbf{A}}$ that can also be written as

$$\lambda_n(\mathbf{A}) = \sup_{v_1,\dots,v_{n-1}\in L^2} \inf_{\substack{u\in \mathrm{Vect}(v_1,\dots,v_{n-1})^\perp \\ \|u\|_{L^2}=1}} \langle Hu, u\rangle.$$

5.2.3 – Resolvent-limit of the renormalised operator

Since the intersection of domains of H and H_ε is trivial, the natural convergence of H_ε to H is in the resolvant sense, this is the following Proposition. In particular, this result explains why our operator H is natural since the regularised operator satisfies

$$\big(i\partial_1 + A_1^{(\varepsilon)}\big)^2 + \big(i\partial_2 + A_2^{(\varepsilon)}\big)^2 = H + c_\varepsilon + o_{\varepsilon\to 0}(1)$$

in the norm resolvent sense.

Proposition 5.11. *Let $\delta > 0$ and $s \in (0,1)$ such that $s < s_\beta(\mathbf{A})$. Then for any constant $k > m_\delta^1(\mathbf{A}, s)$, we have*

$$\|(H+k)^{-1} - (H_\varepsilon + k)^{-1}\|_{L^2\to L^2} \lesssim_{\mathbf{A},s} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha}.$$

Proof: Let $v \in L^2$. Since $H + k : \mathcal{D}_{\mathbf{A}} \to L^2$ is invertible, there exists $u \in \mathcal{D}_{\mathbf{A}}$ such that

$$v = (H+k)u$$

thus

$$\|(H+k)^{-1}v - (H_\varepsilon+k)^{-1}v\|_{L^2} = \|u - (H_\varepsilon + k)^{-1}(H+k)u\|_{L^2}.$$

We introduce $u_\varepsilon := \Gamma_\varepsilon \Phi^s(u)$ which converges to u in L^2 and we have

$$\|u - (H_\varepsilon+k)^{-1}(H+k)u\|_{L^2} \leq \|u - u_\varepsilon\|_{L^2} + \|u_\varepsilon - (H_\varepsilon+k)^{-1}(H+k)u\|_{L^2}.$$

Since Lemma 5.2 gives

$$\|u - u_\varepsilon\|_{L^2} \lesssim_{\mathbf{A},s} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha},$$

we only have to bound the second term. We have

$$\begin{aligned}
\|u_\varepsilon - (H_\varepsilon+k)^{-1}(H+k)u\|_{L^2} &= \|(H_\varepsilon+k)^{-1}\big((H_\varepsilon+k)u_\varepsilon - (H+k)u\big)\|_{L^2} \\
&\lesssim \|(H_\varepsilon+k)u_\varepsilon - (H+k)u\|_{L^2} \\
&\lesssim \|H_\varepsilon u_\varepsilon - Hu\|_{L^2} + k\|u_\varepsilon - u\|_{L^2}
\end{aligned}$$

using Proposition 5.8. In the end, we have

$$\|(H+k)^{-1}v-(H_\varepsilon+k)^{-1}v\|_{L^2}\lesssim\|u_s^\sharp\|_{\mathcal{H}^2}\|\mathbf{A}-\mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha}$$

hence the result since $(H+k)^{-1}:L^2\to\mathcal{D}_\mathbf{A}$ is continuous.

□

5.2.4 – Comparison between the spectrum of *H* and *L*

The following Proposition provides sharp bounds for the eigenvalues of H from the eigenvalues of L, we denote by $(\lambda_n)_{n\ge1}$ the non-decreasing positive sequence of the eigenvalues of L since it corresponds to the case $\mathbf{A}=0$. For $\delta\in(0,1)$ and $s\in(0,1)$, introduce the constant

$$m_\delta^+(\mathbf{A},s):=(1+\delta)(1+ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}).$$

For $s<s_0(\mathbf{A})$, we also introduce

$$m_\delta^-(\mathbf{A},s):=\frac{1-\delta}{1-ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}.$$

Recall $\beta=\frac12(\frac43+2\alpha)$. Write $a,b\le c$ to mean that we have both $a\le c$ and $b\le c$.

Proposition 5.12. *Let $\delta\in(0,1)$ and $s\in(0,1)$ such that $s<s_\beta(\mathbf{A})\wedge s_{1-\frac{\alpha}{4}}(\mathbf{A})$. Given any $n\in\mathbb{Z}^+$, we have*

$$\lambda_n(\mathbf{A}),\lambda_n(\mathbf{A}_\varepsilon)\le m_\delta^+(\mathbf{A},s)\lambda_n+2+2ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}+m_\delta^2(\mathbf{A},s).$$

If moreover $s<s_0(\mathbf{A})$, we have

$$\lambda_n(\mathbf{A}),\lambda_n(\mathbf{A}_\varepsilon)\ge m_\delta^-(\mathbf{A},s)\lambda_n-m_\delta^1(\mathbf{A},s).$$

Proof: Let $u_1^\sharp,\dots,u_n^\sharp\in\mathcal{H}^2$ be an orthonormal family of eigenfunctions of L associated to $\lambda_1,\dots,\lambda_n$ and consider

$$u_i:=\Gamma u_i^\sharp\in\mathcal{D}_\mathbf{A}$$

for $1\le i\le n$. Since Γ is invertible, the family $(u_1,\dots,u_n)$ is free thus the min-max representation of $\lambda_n(\mathbf{A})$ yields

$$\lambda_n(\mathbf{A})\le\sup_{\substack{u\in\mathrm{Vect}(u_1,\dots,u_n)\\ \|u\|_{L^2}=1}}\langle Hu,u\rangle.$$

Given any normalised $u\in\mathrm{Vect}(u_1,\dots,u_n)$, we have

$$\langle Hu,u\rangle\le\|Hu\|_{L^2}\le(1+\delta)\|u_s^\sharp\|_{\mathcal{H}^2}+m_\delta^2(\mathbf{A},s)$$

for $u_s^\sharp=\Phi^s(u)$ using Proposition 5.5 and $s<s_\beta(\mathbf{A})$. Moreover

$$\|u_s^\sharp\|_{\mathcal{H}^2}\le(1+\lambda_n)\|u_s^\sharp\|_{L^2}\le(1+\lambda_n)\Big(1+ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}\Big)$$

hence the upper bound

$$\lambda_n(\mathbf{A})\le m_\delta^+(\mathbf{A},s)\lambda_n+2+2ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}+m_\delta^2(\mathbf{A},s).$$

For the lower bound, we use the min-max representation of $\lambda_n(\mathbf{A})$ under the form

$$\lambda_n(\mathbf{A}) = \sup_{v_1,\dots,v_{n-1}\in L^2} \inf_{\substack{u\in \mathrm{Vect}(v_1,\dots,v_{n-1})^\perp \\ \|u\|_{L^2}=1}} \langle Hu, u\rangle.$$

Introducing

$$F := \mathrm{Vect}(u_m; m \geq n),$$

we have that $F^\perp$ is a subspace of L^2 of finite dimension $n-1$ thus there exists a orthogonal family $(v_1,\dots,v_{n-1})$ such that $F^\perp = \mathrm{Vect}(v_1,\dots,v_{n-1})$. Since F is a closed subspace of L^2 as an intersection of hyperplans, we have $F = \mathrm{Vect}(v_1,\dots,v_{n-1})^\perp$ hence

$$\lambda_n(\mathbf{A}) \geq \inf_{\substack{u\in F \\ \|u\|_{L^2}=1}} \langle Hu, u\rangle.$$

Let $u \in F$ with $\|u\|_{L^2} = 1$. Using Proposition 5.7, we have

$$\begin{aligned}
\langle Hu, u\rangle &\geq (1-\delta)\langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle - m_\delta^1(\mathbf{A}, s) \\
&\geq (1-\delta)\langle u_s^\sharp, Lu_s^\sharp\rangle - m_\delta^1(\mathbf{A}, s) \\
&\geq (1-\delta)\lambda_n \|u_s^\sharp\|_{L^2}^2 - m_\delta^1(\mathbf{A}, s).
\end{aligned}$$

Finally using Proposition 5.1 for $s < s_{1-\frac{\alpha}{4}}(\mathbf{A})$, we get

$$\langle Hu, u\rangle \geq \frac{1-\delta}{1 - ms^{\frac{\alpha}{2}}\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\lambda_n - m_\delta^1(\mathbf{A}, s)$$

and the proof is complete.

$\square$

In particular, taking

$$s = \left(\frac{\delta}{m\|\mathbf{A}\|_{\mathcal{X}^\alpha}}\right)^{\frac{2}{\alpha}}$$

gives the simpler bounds

$$\lambda_n - m_\delta^1(\mathbf{A}) \leq \lambda_n(\mathbf{A}) \leq (1+\delta)^2\lambda_n + m_\delta^2(\mathbf{A})$$

for any δ small enough. This is sharp enough to get an almost sure Weyl-type law from the Weyl law for L.

Corollary 5.13. *We have*

$$\lim_{\lambda\to\infty} \lambda^{-1}|\{n \geq 1; \lambda_n(\mathbf{A}) \leq \lambda\}| = \pi$$

Proof : The lower and upper bounds on the eigenvalues give

$$N\left(\frac{\lambda - m_\delta^2(\mathbf{A})}{1+\delta}\right) \leq \big|\{n \geq 1; \lambda_n(\mathbf{A}) \leq \lambda\}\big| \leq N\big(\lambda + m_\delta^1(\mathbf{A})\big)$$

hence the proof is complete using the result for the Laplacian.

$\square$

5.3 – Renormalisation and enhanced potential

The enhanced potential

$$\mathbf{A} := (A, A^2) \in \mathcal{X}^\alpha = \mathcal{C}^{\alpha-1} \times \mathcal{C}^{2\alpha-2}$$

is defined with $A^2 := A \cdot A$ for regular enough potential, $A \in L^\infty$ is enough. For arbitrary distributions $A \in \mathcal{C}^{\alpha-1}$, this does not make sense since the product of two distributions in Hölder spaces can be defined only if the sum of their regularity exponents is positive. The magnetic Laplacian with white noise $B = \xi$ as magnetic field corresponds to this framework since the associate magnetic potential

$$A := \nabla^\perp \Phi$$

where $\Phi = \Delta^{-1}\xi$ belongs to $\mathcal{C}^{\alpha-1}$ for any $\alpha < 1$. In this case, one has to make sense of

$$A \cdot A = (-\partial_2 \Phi)^2 + (\partial_1 \Phi)^2$$

which is expected to belong to $\mathcal{C}^{2\alpha-2}$ since $\alpha - 1 < 0$. A natural way of procedding is to consider a regularisation of the noise $\xi_\varepsilon := \xi * \rho_\varepsilon$. In this case, the associated magnetic potential A_ε is smooth and the product $A_\varepsilon \cdot A_\varepsilon$ is well-defined. The singularity of the limit translates as the almost sure divergence of the product as ε goes to 0. Indeed for $x \in \mathbb{T}^2$, one has

$$\begin{aligned}
\mathbb{E}\big[A_\varepsilon(x) \cdot A_\varepsilon(x)\big[\big] &= \mathbb{E}\big[(\partial_2 \Delta^{-1}\xi_\varepsilon)^2(x) + (\partial_1 \Delta^{-1}\xi_\varepsilon)^2(x)\big] \\
&= \big\|\partial_2 G_\varepsilon(x,\cdot)\big\|_{L^2}^2 + \big\|\partial_1 G_\varepsilon(x,\cdot)\big\|_{L^2}^2
\end{aligned}$$

with G the Green function of the Laplacian and $G_\varepsilon(x,\cdot) := G(x,\cdot) * \rho_\varepsilon$. Hence the mean of the regularised product diverges as

$$c_\varepsilon := \mathbb{E}[A_\varepsilon(0) \cdot A_\varepsilon(0)] \underset{\varepsilon \to 0}{\sim} \frac{\ln(\varepsilon)}{4\pi^2}.$$

While the mean of the random variable $A_\varepsilon(x) \cdot A_\varepsilon(x)$ diverges, one can try to describe the fluctuation around this asymptotic and find a limit to

$$A_\varepsilon \cdot A_\varepsilon - \mathbb{E}[A_\varepsilon \cdot A_\varepsilon]$$

as ε goes to 0. It happens that this converges to a limit in the expected Hölder space $\mathcal{C}^{2\alpha-2}$, this is the Wick product. As far as discrete associated models are concerned, this can be interpreted as a central limit Theorem where one describes the fluctuation around a diverging number of particules, see for example [47] for an example with the Anderson Hamiltonian.

Theorem 5.14. *There exists a random distribution A^2 that belongs almost surely to $\mathcal{C}^{2\alpha-2}$ such that*

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\big\|A^2 - (A_\varepsilon \cdot A_\varepsilon - c_\varepsilon)\big\|_{\mathcal{C}^{2\alpha-2}}^p\Big] = 0$$

for any $p \geq 1$.

Proof : Since the noise is Gaussian, we only need to control second order moment using hypercontractivity. The resonant term $\mathsf{\Pi}(A_\varepsilon, A_\varepsilon)$ is a linear combination of terms of the form

$$I_\varepsilon := \int_0^1 P_t\left(Q_t^1 A_\varepsilon \cdot Q_t^2 A_\varepsilon\right) \frac{\mathrm{d}t}{t}$$

with $P \in \mathsf{StGC}^{[0,b]}$ and $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$. We also define the renormalised quantity

$$J_\varepsilon := I_\varepsilon - \mathbb{E}[I_\varepsilon].$$

Let $u \in (0,1), x \in \mathbb{T}^2$ and $Q \in \mathsf{StGC}^r$ with $r > |2\alpha - 2|$. The expectation $\mathbb{E}\left[|Q_u\big(I_\varepsilon\big)(x)|^2\right]$ is given by the integral over $\mathbb{T}^2 \times \mathbb{T}^2 \times [0,1]^2$ of

$$K_{Q_u P_t}(x,y) K_{Q_u P_s}(x,z) \mathbb{E}\Big[Q_t^1 A_\varepsilon(y) Q_t^2 A_\varepsilon(y) Q_s^1 A_\varepsilon(z) Q_s^2 A_\varepsilon(z)\Big]$$

against the measure $\mu(\mathrm{d}y)\mu(\mathrm{d}z)(ts)^{-1}\mathrm{d}t\mathrm{d}s$. Using the Wick formula, we have

$$\begin{aligned}
\mathbb{E}\Big[Q_t^1 A_\varepsilon(y) Q_t^2 A_\varepsilon(y) Q_s^1 A_\varepsilon(z) Q_s^2 A_\varepsilon(z)\Big] &= \mathbb{E}\left[Q_t^1 A_\varepsilon(y) Q_t^2 A_\varepsilon(y)\right] \mathbb{E}\left[Q_s^1 A_\varepsilon(z) Q_s^2 A_\varepsilon(z)\right] \\
&\quad + \mathbb{E}\left[Q_t^1 A_\varepsilon(y) Q_s^1 A_\varepsilon(z)\right] \mathbb{E}\left[Q_t^2 A_\varepsilon(y) Q_s^2 A_\varepsilon(z)\right] + \mathbb{E}\left[Q_t^1 A_\varepsilon(y) Q_s^2 A_\varepsilon(z)\right] \mathbb{E}\left[Q_s^1 A_\varepsilon(z) Q_t^2 A_\varepsilon(y)\right] \\
&= (1) + (2) + (3)
\end{aligned}$$

and this yields

$$\mathbb{E}\left[|Q_u\big(I_\varepsilon\big)(x)|^2\right] = I_\varepsilon^{(1)}(x) + I_\varepsilon^{(2)}(x) + I_\varepsilon^{(3)}(x).$$

The first term corresponds exactly to the extracted diverging quantity since

$$I_\varepsilon^{(1)} = \mathbb{E}\left[\int_0^1 Q_u P_t^\bullet\left(Q_t^1 A_\varepsilon \cdot Q_t^2 A_\varepsilon\right) \frac{\mathrm{d}t}{t}\right]^2 = \mathbb{E}\big[Q_u(I_\varepsilon)\big]^2$$

and we have

$$\mathbb{E}\left[|Q_u\big(J_\varepsilon\big)(x)|^2\right] = \mathbb{E}\left[\Big\{Q_u\big(I_\varepsilon\big)(x) - \mathbb{E}[Q_u\big(I_\varepsilon\big)](x)\Big\}^2\right] = I_\varepsilon^{(2)}(x) + I_\varepsilon^{(3)}(x).$$

Using that $(\Psi(\varepsilon L))_\varepsilon$ belongs to G, ξ is an isometry from L^2 to square-integrable random variables, we have

$$\begin{aligned}
&I_\varepsilon^{(2)}(x) + I_\varepsilon^{(3)}(x) \\
&\quad\lesssim \int_{\mathbb{T}^2\times\mathbb{T}^2} \int_{[0,1]^2} K_{Q_u P_t^\bullet}(x,y) K_{Q_u P_s^\bullet}(x,z) \big\langle \mathcal{G}_{2\varepsilon+t+s}(y,\cdot), \mathcal{G}_{2\varepsilon+t+s}(z,\cdot)\big\rangle^2 \mu(\mathrm{d}y)\mu(\mathrm{d}z) ts\mathrm{d}t\mathrm{d}s \\
&\quad\lesssim \int_{\mathbb{T}^2\times\mathbb{T}^2} \int_{[0,1]^2} K_{Q_u P_t^\bullet}(x,y) K_{Q_u P_s^\bullet}(x,z) \mathcal{G}_{2\varepsilon+t+s}(y,z)^2 \mu(\mathrm{d}y)\mu(\mathrm{d}z) ts\mathrm{d}t\mathrm{d}s \\
&\quad\lesssim \int_{\mathbb{T}^2\times\mathbb{T}^2} \int_{[0,1]^2} \mathcal{G}_{u+t}(x,y) \mathcal{G}_{u+s}(x,z) \mathcal{G}_{2\varepsilon+t+s}(y,z)^2 \mu(\mathrm{d}y)\mu(\mathrm{d}z) ts\mathrm{d}t\mathrm{d}s \\
&\quad\lesssim \int_{\mathbb{T}^2\times\mathbb{T}^2} \int_{[0,1]^2} (2\varepsilon+t+s)^{-\frac{d}{2}} \mathcal{G}_{u+t}(x,y) \mathcal{G}_{u+s}(x,z) \mathcal{G}_{2\varepsilon+t+s}(y,z) \mu(\mathrm{d}y)\mu(\mathrm{d}z) ts\mathrm{d}t\mathrm{d}s \\
&\quad\lesssim \int_{[0,1]^2} (2\varepsilon+t+s)^{-\frac{d}{2}} (\varepsilon+u+t+s)^{-\frac{d}{2}} ts\mathrm{d}t\mathrm{d}s \\
&\quad\lesssim (\varepsilon+u)^{2-d}
\end{aligned}$$

hence the family $\big(\Pi(A_\varepsilon, A_\varepsilon) - c_\varepsilon\big)_{\varepsilon>0}$ is bounded in $\mathcal{C}^{2\alpha-2}$ for any $\alpha < 1$ since $d = 2$. These computations also show that the associated linear combination of

$$J := \int_0^1 \Big\{ P_t^\bullet \left(Q_t^1 A \cdot Q_t^2 A\right) - \mathbb{E}\left[P_t^\bullet \left(Q_t^1 A \cdot Q_t^2 A\right)\right] \Big\} \frac{\mathrm{d}t}{t}$$

yields a well-defined random distribution of $\mathcal{C}^{2\alpha-2}$ for $\alpha < 1$ that we denote $\Pi(A, A)$. The same type of computations show the convergence and completes the proof with

$$A^2 := 2\mathsf{P}_A A + \Pi(A, A).$$

$\square$

Then $\mathbf{A} := (A, A^2)$ belongs to $\mathcal{X}^\alpha$ and

$$\lim_{\varepsilon\to 0} \|\mathbf{A} - \mathbf{A}_\varepsilon\|_{\mathcal{X}^\alpha} = 0$$

with

$$\mathbf{A}_\varepsilon := (A_\varepsilon, A_\varepsilon \cdot A_\varepsilon - c_\varepsilon) \in \mathcal{X}^\alpha.$$

Remark :

- *The effect of a change of gauge $\widetilde{A} = A + \mathrm{d}f$ on H can be seen at the level of the regularised operator. It would give*

$$\widetilde{H}_\varepsilon = (i\partial_1 + \widetilde{A}_1)^2 + (i\partial_2 + \widetilde{A}_2)^2 - \tilde{c}_\varepsilon$$

  with

$$\widetilde{c}_\varepsilon = \mathbb{E}[\widetilde{A}_\varepsilon \cdot \widetilde{A}_\varepsilon].$$

  Since the spectral properties of the magnetic Laplacian with smooth potential are not affected by the choice of gauge, it only remains to see the impact in the renormalisation procedure. It gives

$$\widetilde{c}_\varepsilon = \mathbb{E}[A_\varepsilon \cdot A_\varepsilon] + 2\mathbb{E}[\mathrm{d}f \cdot A_\varepsilon] + \mathbb{E}[\mathrm{d}f \cdot \mathrm{d}f] = c_\varepsilon + |\mathrm{d}f|^2$$

  in the case of a deterministic change of gauge $\mathrm{d}f$. This would change the spectral properties of the limit however one could recover the same operator by incorporating the term $|\mathrm{d}f|^2$ in the renormalisation procedure since it does not cause any divergence. The arbitrary choice one has to make in the renormalisation allows to deal with different gauge choice and should be motivated by the applications.

- *As for the Anderson Hamiltonian, this raises interesting questions as far as probability is concerned. For example, the eingenvalues are random variables and one could get tail estimates as in [50]. One could also consider the martingale problem associated to rough differential equations (RDEs) in the case of a time-independant distributional drift, see Chapter 7.*

Chapter 6

Dispersive singular SPDEs

While the previous Chapters dealt with parabolic and elliptic PDEs, substantial progress was naturally also made in the field of singular dispersive SPDEs following the paper [28] due to Debussche and Weber on the cubic multiplicative stochastic Schrödinger equation and the paper [36] by Gubinelli, Koch and Oh on the cubic additive stochastic wave equation. Since the powerful tools from singular SPDEs are only directly applicable to parabolic and elliptic SPDEs, these initial papers were in a not so singular regime, the former using an exponential transform to remove the most singular term and the latter using a "DaPrato-Debussche trick" to do the same. In [38], Gubinelli, Ugurcan and Zachhuber proved some sharpened results on the multiplicative Schrödinger equation and its wave analogue by reframing it in relation to the Anderson Hamiltonian as well as extending the results to dimension 3. Building on this, Strichartz estimates were shown to hold in [59] by Zachhuber for the Anderson Hamiltonian which, in a nutshell, leverage dispersion in order to allow to trade integrability in time for integrability in space, see Section 6.1.2 for a more detailed introduction. Moreover, Tzvetkov and Visciglia extended in [58] the results of [28] to a larger range of power nonlinearities. For the nonlinear wave equation with additive noise, let us mention here the follow-up paper by Gubinelli, Koch and Oh [37] in three dimensions with quadratic nonlinearity and the paper [52] by Oh, Robert and Tzvetkov which extends the results of [36] to the case of two-dimensional surfaces and is thus salient for the current Chapter.

In this Chapter, we first explain how the construction of H yields immediately to strong energy solutions for the cubic nonlinear Schrödinger equation with multiplicative noise on a two-dimensional manifold while the form estimate gives energy solution. We then prove Strichartz inequalities for the Schrödinger and wave equation with white noise potential on compact surfaces. Moreover, we show how this provides local well-posedness for the associated nonlinear equations in a low-regularity regimes. As for the deterministic case, the Strichartz estimates obtained depend whether the manifold has a boundary or not and are improved in the flat case of the torus. By Strichartz inequalities, we generally refer to space-time bounds on the propagators of Schrödinger and wave equations where the results on integrability are strictly better than what one gets from the Sobolev embedding so – for definiteness we consider the Schrödinger case – a bound like

$$\|e^{itH}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha},$$

with $p \in [1,\infty], q > \frac{2d}{d-2\alpha}$ where d denotes the dimension and $I \subset \mathbb{R}$ is an interval.

The overall approach to the Schrödinger group associated to H we follow is similar to the one in [59], where such Strichartz estimates were shown for the Anderson Hamiltonian on the two and three-dimensional torus. However, one gets sharper results in the particular case of flat geometry due to the fact that one has stronger classical Strichartz inequalities available. In the more general setting of a Riemannian compact manifold, we work with a result due to Burq, Gérard and Tzvetkov [19] which has been extended to the case with boundary by Blair, Smith and Sogge in [13]. These results can be thought of as quantifying the statement "finite frequencies travel at finite speeds – in (frequency dependent) short time the evolution is morally on flat space". Let us also mention at this point the recent work by Huang and Sogge [41] which deals with a similar setting, however their notion of singular potential refers to low integrability while in our case singular refers rather to potentials with low regularity.

For the case of Strichartz estimates for the wave equation related to H, we follow the approach introduced by Burq, Lebeau and Planchon [20] on domains with boundary. The main idea, which is why this approach is applicable, is that all that is required is that the operator driving the wave equation satisfies some growth condition on the L^q bounds on the its eigenfunctions and one knows about the asymptotics of the eigenvalues, in their case the Laplace with boundary conditions. Since a Weyl law for H was obtained by Mouzard in [50] and our result for the Schrödinger equation gives us a suitable L^q bound on the eigenfunctions of H, their approach turns out to be enough to prove Strichartz estimates that beat the Sobolev embedding. One also gets improved results in the flat case of the two-dimensional torus since one has sharper L^q bounds coming from the sharper Strichartz estimates proved by Zachhuber in [59]. Overall this approach seems somewhat crude and we assume there to be sharper bounds possible whereas in the Schrödinger case, our result is the same as the one without noise obtained in [19] worsened only by an arbitrarily small regularity. The state of the art of Strichartz estimates for wave equations on manifolds with boundary is the paper [12], the case of manifolds without boundary being comparable to the Strichartz estimates on Euclidean space because of the finite speed of propagation. The final objective of this Chapter is to use the Strichartz inequalities obtained to prove local well-posedness for the associated defocussing nonlinear equations, also known as cubic multiplicative stochastic Schrödinger and wave equations. This will be done using fairly straightforward contraction arguments for which the Strichartz estimates will be crucial, this is analogous to Section 5 of [59]. The results of this Chapter are from the works [50, 51].

6.1 – Nonlinear Schrödinger equation

In the first Section, we explain how the construction of the Anderson Hamiltonian H and its form domain yields strong and energy solutions. We then provide Strichartz inequalities for the Schrödinger groups asscoiated to H after recalling the proof of this result for the Laplace-Beltrami operator from [19]. Finally, we show how this can be used to get local well-posedness in low-regularity Sobolev spaces. Recall that H is constructed as the limit of

$$H_\varepsilon := L + \xi_\varepsilon - c_\varepsilon$$

with c_ε a diverging function as ε goes to 0. In particular, one can take shift c_ε by a large enough constant to ensure that H is positive.

6.1.1 – Strong and energy solutions

The construction of the Anderson Hamiltonian allows the study of associated evolution equations. This was the motivation for the work [38] of Gubinelli, Ugurcan and Zachhuber and they studied the nonlinear Schrödinger and wave equations on the torus in two and three dimensions. The construction from Chapter 4 allows to do the same on a two-dimensional manifold. As an example, we give results for the cubic nonlinear Schrödinger equation associated to H. See the work [27] of Debussche and Weber for the equation on the torus where they use a Hopf-Cole type transformation. This was extended in [58] by Tzvetkov and Visciglia to the fourth order nonlinearity. Since H is positive, Proposition 5.9 yields a characterization of the domain and the form domain which is defined as follows.

Definition. *We define the form domain of H denoted $\mathcal{D}_\Xi(\sqrt{H})$ as the closure of the domain under the norm*

$$\|u\|_{\mathcal{D}_\Xi(\sqrt{H})} := \sqrt{\langle u, Hu\rangle}$$

In particular, the form estimate on H gives the following parametrisation of the form domain using the Γ map in addition to the parametrisation of the domain $\mathcal{D}_\Xi$.

Proposition 6.1. *For $s < s_0(\Xi)$ and $u \in L^2$,*

$$\big(u \in \mathcal{D}_\Xi(\sqrt{H})\big) \iff \big(\Phi^s(u) = u_s^\sharp \in \mathcal{H}^1\big)$$

with the bounds

$$\|u_s^\sharp\|_{\mathcal{H}^1} \lesssim_{\Xi,s} \|u\|_{\mathcal{D}_\Xi(\sqrt{H})} \lesssim_{\Xi,s} \|u_s^\sharp\|_{\mathcal{H}^1}.$$

Proof: As stated, Proposition 4.9 yields

$$\|u_s^\sharp\|_{\mathcal{H}^1} \lesssim_{\Xi,s} \|u\|_{\mathcal{D}_\Xi(\sqrt{H})}.$$

In fact, the inequality that is proved is

$$\big|\langle Hu, u\rangle - \langle \nabla u_s^\sharp, \nabla u_s^\sharp\rangle\big| \leq k_\Xi \|u\|_{L^2} + \delta \|u_s^\sharp\|_{\mathcal{H}^1}$$

thus one also get the other estimate

$$\|u\|_{\mathcal{D}_\Xi(\sqrt{H})} \lesssim_{\Xi,s} \|u_s^\sharp\|_{\mathcal{H}^1}.$$

$\square$

This yields a version of Brezis-Gallouët inequality for the Anderson Hamiltonian. In some sense, it interpolates the L^∞-norm between the energy norm and the logarithm of the domain norm. This was already obtained in [38] by Gubinelli, Ugurcan and Zachhuber on the torus.

Theorem 6.2. *For any $v \in \mathcal{D}_\Xi$, we have*

$$\|v\|_{L^\infty} \lesssim_\Xi \|v\|_{\mathcal{D}_\Xi(\sqrt{H})} \left(1 + \sqrt{\log\left(1 + \frac{\|v\|_{\mathcal{D}_\Xi}}{\|v\|_{\mathcal{D}(\sqrt{H})}}\right)}\right).$$

For any $v \in \mathcal{H}^2$, we have

$$\|v\|_{L^\infty} \lesssim_\Xi \|\sqrt{H_\varepsilon} v\|_{L^2} \left(1 + \sqrt{\log\left(1 + \frac{\|H_\varepsilon v\|_{L^2}}{\|\sqrt{H_\varepsilon} v\|_{L^2}}\right)}\right).$$

In particular, the second inequality holds uniformly in ε.

Proof : For any $t > 0$, we have

$$v = \int_0^t (t'L)e^{-t'L}v\frac{\mathrm{d}t'}{t'} + e^{-tL}v$$

thus

$$\|v\|_{L^\infty} \le \left\|\int_0^t (t'L)e^{-t'L}v\frac{\mathrm{d}t'}{t'}\right\|_{L^\infty} + \|e^{-tL}v\|_{L^\infty}.$$

One can bound the integral as

$$\begin{aligned}\left\|\int_0^t (t'L)e^{-t'L}v\frac{\mathrm{d}t'}{t'}\right\|_{L^\infty} &\lesssim \int_0^t \|Lv\|_{L^2}\mathrm{d}t' \\ &\lesssim t\|v\|_{\mathcal{H}^2}\end{aligned}$$

and the remainder as

$$\begin{aligned}\|e^{-tL}v\|_{L^\infty} &\lesssim \left\|\int_t^1 (t'L)e^{-t'L}v\frac{\mathrm{d}t'}{t'}\right\|_{L^\infty} + \|e^{-L}v\|_{L^\infty} \\ &\lesssim \left(\int_t^1 \frac{\mathrm{d}t'}{t'}\right)^{\frac12}\left(\int_t^1 \|(t'L)e^{-t'L}v\|_{L^\infty}^2\frac{\mathrm{d}t'}{t'}\right)^{\frac12} + \|v\|_{\mathcal{H}^1} \\ &\lesssim \left(\int_t^1 \frac{\mathrm{d}t'}{t'}\right)^{\frac12}\left(\int_t^1 \|\sqrt{t'}Le^{-t'L}v\|_{L^2}^2\mathrm{d}t'\right)^{\frac12} + \|v\|_{\mathcal{H}^1} \\ &\lesssim \|v\|_{\mathcal{H}^1}\big(1+|\log(t)|^{\frac12}\big),\end{aligned}$$

to get

$$\|v\|_{L^\infty} \lesssim t\|v\|_{\mathcal{H}^2} + \big(1+|\log(t)|^{\frac12}\big)\|v\|_{\mathcal{H}^1}.$$

Taking $\|v\|_{\mathcal{H}^1} \le 1$ and $t = \frac{\sqrt{\log(1+\|v\|_{\mathcal{H}_2})}}{1+\|v\|_{\mathcal{H}^2}} > 0$, we get the classical Brezis-Gallouet inequality, that is

$$\|v\|_{L^\infty} \lesssim 1 + \sqrt{\log\left(1+\|v\|_{\mathcal{H}^2}\right)}.$$

Thus for $\|v\|_{\mathcal{D}(\sqrt{H})} \le 1$, we have

$$\begin{aligned}\|v\|_{L^\infty} &\lesssim_\Xi \|v^\sharp\|_{L^\infty} \\ &\lesssim_\Xi 1 + \sqrt{\log\left(1+\|v^\sharp\|_{\mathcal{H}^2}\right)} \\ &\lesssim_\Xi 1 + \sqrt{\log\big(1+\|H^+\|_{\mathcal{D}(H)}\big)}\end{aligned}$$

using Proposition 6.1. Since every estimates also hold for H_ε with bound uniform in ε, we also get the estimate for H_ε. Applying this result to $\frac{v}{\|v\|_{\mathcal{D}(\sqrt{H})}}$ yields the general inequality.

$\square$

This inequality can be used for example to study the cubic nonlinear Schrödinger equation with multiplicative noise

$$i\partial_t u - \Delta u + u\xi = -|u|^2u$$

with initial condition $u_0 \in \mathcal{D}_\Xi$. The construction of the operator H immediatly yields the renormalised solution $u(t,\cdot) := e^{-itH}u_0$ to the linear equation

$$i\partial_t u - \Delta u + u\xi = 0$$

given any $u_0 \in \mathcal{D}_\Xi$. This is the content of the following Theorem. Remark that when one regularizes the question, one also has to consider a suitable sequence of initial data $(u_0^{(\varepsilon)})_{\varepsilon>0}$, it is often refered to as "well-prepared data" in the litterature.

Theorem 6.3. *Let $T > 0$ and $u_0 \in \mathcal{D}_\Xi$. Then there exists a unique solution $u \in C\big([0,T],\mathcal{D}_\Xi\big) \cap C^1\big([0,T],L^2\big)$ to the equation*

$$\left\{\begin{array}{rcl} i\partial_t u &=& Hu \\ u(0,\cdot) &=& u_0 \end{array}\right. \qquad \text{on } [0,T]\times M.$$

Moreover, u is the L^2-limit of the solutions $u_\varepsilon \in C\big([0,T],\mathcal{H}^2\big) \cap C^1\big([0,T],L^2\big)$ of solutions to the equations

$$\left\{\begin{array}{rcl} i\partial_t u_\varepsilon &=& H_\varepsilon u_\varepsilon \\ u_\varepsilon(0,\cdot) &=& u_0^{(\varepsilon)} \end{array}\right. \qquad \text{on } [0,\infty[\times M,$$

with the initial data

$$u_0^{(\varepsilon)} := (H_\varepsilon)^{-1}Hu_0 \in \mathcal{H}^2$$

which converges to u_0 in L^2.

One can also solve the associated equation with cubic nonlinearity. One can not apply the same Theorem as Brezis and Gallouët in [15] since we do not have a control on the cubic term from $\mathcal{D}_\Xi$ to itself. One could modify the domain taking into account the term $\Pi(X_1,X_1)$ in X_2 to get a domain stable by multiplication. However since a direct computation as done by Gubinelli, Ugurcan and Zachhuber in [38] is enough, it is not necessary. In particular, the proof of the following Theorem works exactly as in their work and is left to the reader. This will be detailled in Section 6.1.5 with the use of Strichartz inequalities.

Theorem 6.4. *Let $T > 0$ and $u_0 \in \mathcal{D}_\Xi$. Then there exists a unique solution $u \in C\big([0,T],\mathcal{D}_\Xi\big) \cap C^1\big([0,T],L^2\big)$ to the equation*

$$\left\{\begin{array}{rcl} i\partial_t u &=& Hu - |u|^2u \\ u(0,\cdot) &=& u_0 \end{array}\right. \qquad \text{on } [0,T]\times M.$$

Moreover, u is the L^2-limit of the solutions $u_\varepsilon \in C\big([0,T],\mathcal{H}^2\big) \cap C^1\big([0,T],L^2\big)$ of solutions to the equations

$$\left\{\begin{array}{rcl} i\partial_t u_\varepsilon &=& H_\varepsilon u_\varepsilon - |u_\varepsilon|^2u_\varepsilon \\ u_\varepsilon(0,\cdot) &=& u_0^{(\varepsilon)} \end{array}\right. \qquad \text{on } [0,\infty[\times M,$$

with the initial data

$$u_0^{(\varepsilon)} := (H_\varepsilon)^{-1}Hu_0 \in \mathcal{H}^2$$

which converges to u_0 in L^2. We also have the convergences

$$\begin{aligned} u_\varepsilon(t) &\to u(t) \quad \text{in } L^2, \\ H_\varepsilon u_\varepsilon(t) &\to Hu(t) \quad \text{in } L^2, \\ \partial_t u_\varepsilon(t) &\to \partial_t u(t) \quad \text{in } L^2 \end{aligned}$$

for all $t \in [0,T]$.

Remark : *The solution to*

$$i\partial_t v_\varepsilon = \Delta v_\varepsilon + \xi_\varepsilon v_\varepsilon - |v_\varepsilon|^2 v_\varepsilon$$

on the torus can be related to the renormalised equation

$$i\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon - |u_\varepsilon|^2 u_\varepsilon$$

via the change of variable $u_\varepsilon(t,\cdot) = e^{tc_\varepsilon} v_\varepsilon(t,\cdot)$ since c_ε is a constant as Tzvetkov and Visciglia's Theorem 1.1 *from [58]. One could want to do the same in a manifold setting, it is however not clear what the change of variable should be on a manifold since c_ε is a function and not a constant. It should still be possible to find an appropriate change of variable even though this requires some work.*

6.1.2 – Strichartz inequalities for the Laplace-Beltrami operator

Since our proof of the Schrödinger Strichartz inequalities for the torus relies on the result for the deterministic equation, we first explain how one gets these estimates in this case. As explained, we use the result obtained by Burq, Gérard and Tzvetkov in [19]. On the torus, regularity of distributions can be measured using the Littlewood-Paley decomposition. On a manifold, one has an analogue decomposition using the eigenfunctions of the Laplace-Beltrami operator Δ as a generalisation of Fourier theory, see for example Section 2 in [53] by Oh, Robert, Tzvetkov and Wang and references therein. Let (M, g) be a two-dimensional compact Riemannian manifold without boundary or with boundary and Dirichlet boundary conditions. In this framework, the Laplace-Beltrami operator $-\Delta$ is a self-adjoint positive operator with discrete spectrum

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

with the associated normalized eigenfunctions $(\varphi_n)_{n\geq 1}$ belonging to $C^\infty(M)$. Furthermore, the Weyl law gives the asymptotics

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{\mathrm{Vol}(M)}{4\pi}.$$

The basis $(\varphi_n)_{n\geq 1}$ of L^2 gives the decomposition

$$u = \sum_{n\geq 1} \langle u, \varphi_n\rangle \varphi_n$$

for any distribution $u \in \mathcal{D}'(M)$. On the torus, this gives the Littlewood-Paley decomposition of u where the regularity is measured by the asymptotics behavior of $\sum_{\lambda_k\sim 2^n} \langle u, \varphi_k\rangle$. On a manifold M, this is done with

$$\Delta_n := \psi\big(-2^{-2(n+1)}\Delta\big) - \psi\big(-2^{-2n}\Delta\big)$$

for $n \geq 0$ and

$$\Delta_{-1} := \psi(-\Delta)$$

with $\psi \in C_0^\infty(\mathbb{R})$ a non-negative function with $\mathrm{supp}(\psi) \subset [-1, 1]$ and $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Recall that for any function $\psi \in L^\infty(\mathbb{R})$, the operator $\psi(\Delta)$ is defined as

$$\psi(\Delta)u = \sum_{n\geq 1} \psi(\lambda_n)\langle u, \varphi_n\rangle \varphi_n$$

and this yields a bounded operator from $L^2(M)$ to itself. In this setting, Besov spaces can also be defined for $\alpha \in \mathbb{R}$ and $p,q \in [1,\infty]$ as

$$\mathcal{B}^{\alpha}_{p,q} := \{u \in \mathcal{D}'(M)\ ;\ \|u\|_{\mathcal{B}^{\alpha}_{p,q}} < \infty\}$$

where

$$\|u\|_{\mathcal{B}^{\alpha}_{p,q}} := \Big(\|\Delta_{-1}u\|^q_{L^p(M)} + \sum_{n\geq 0} 2^{\alpha q}\|\Delta_n u\|^q_{L^p(M)}\Big)^{\frac{1}{q}}.$$

The particular case $p = q = 2$ corresponds to Sobolev spaces and we have

$$\|u\|^2_{\mathcal{H}^{\alpha}} = \|\Delta_{-1}u\|^2_{L^2(M)} + \sum_{n\geq 0} 2^{2n\alpha}\|\varphi(2^{-2n}\Delta)u\|^2_{L^2(M)}$$

where $\varphi(x) := \psi(-x^2) - \psi(-x)$. Burq, Gérard and Tzvetkov proved in [19] the bound

$$\|f\|_{L^q(M)} \lesssim \|\psi(-\Delta)f\|_{L^q(M)} + \Big(\sum_{n\geq 0}\|\varphi(2^{-2n}\Delta)f\|^2_{L^q(M)}\Big)^{\frac{1}{2}}$$

using that for $\lambda \in \mathbb{R}$, we have

$$\psi(-\lambda) + \sum_{n\geq 0}\varphi(2^{-2n}\lambda) = 1.$$

Applying this to the Schrödinger group, they obtain

$$\|e^{it\Delta}v\|_{L^p([0,1],L^q)} \lesssim \|\psi(-\Delta)v\|_{L^q(M)} + \Big\|\Big(\sum_{k\geq 0}\|e^{it\Delta}\varphi(2^{-2k}\Delta)v\|^2_{L^q(M)}\Big)^{\frac{1}{2}}\Big\|_{L^p([0,1])}$$

using that the Paley-Littlewood projectors commute with the Schrödinger group hence one only needs a bound for spectrally localised data. This is proved using semi-classical analysis with the use of the WKB expansion, see Proposition 2.9 from [19] and references therein which gives

$$\Big(\int_J \|e^{it\Delta}\varphi(h^2\Delta)v\|^p_{L^q(M)}\mathrm{d}t\Big)^{\frac{1}{p}} \lesssim \|v\|_{L^2(M)} \tag{6.1}$$

for J an interval of small enough length proportional to $h \in (0,1)$. Moreover, a well-known trick is to slice up the time interval into small pieces, this will be useful later. The previous bounds with the Minkowski inequality lead to

$$\|e^{it\Delta}v\|_{L^p([0,1],L^q)} \lesssim \|v\|_{L^2(M)} + \Big(\sum_{k\geq 0} 2^{2k/p}\|\varphi(2^{-2k}\Delta)v\|^2_{L^2(M)}\Big)^{\frac{1}{2}} \lesssim \|v\|_{\mathcal{H}^{\frac{1}{p}}}.$$

This yields the following Theorem.

Theorem 6.5. *Let $p \geq 2$ and $q < \infty$ such that*

$$\frac{2}{p} + \frac{2}{q} = 1.$$

Then

$$\|e^{-it\Delta}u\|_{L^p([0,1],L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}}}.$$

While this result is optimal on general surfaces, this can be improved in the flat case of the torus. For the Anderson Hamiltonian, analogue result was obtained by Zachhuber in [59] and we also expect the bound to be weaker on manifolds. This is indeed the case and we obtained an the same result with an arbitrary loss of regularity, this is the content of Section 6.1.4. In the case of a manifold with boundary, the following result was obtained by Blair, Smith and Sogge [13].

Theorem 6.6. *Let $p \in (2, \infty]$ and $q \in [2, \infty)$ such that*

$$\frac{3}{p} + \frac{2}{q} \leq 1$$

and consider σ given by

$$\frac{2}{p} + \frac{2}{q} = 1 - \sigma.$$

Then

$$\|e^{-it\Delta} u\|_{L^p([0,1],L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}+\sigma}}.$$

We end this Section with two classical results that will be needed in this paper. First, one still has Bernstein Lemma with the Littlewood-Paley decomposition associated to the Laplace-Beltrami operator.

Lemma 6.7. *Let $g : M \to \mathbb{R}$ be a function which has spectral support in an interval $[a, b]$ with $0 < a < b < \infty$. Then for any $\alpha, \beta \in \mathbb{R}$ we have the following bounds which are the analogue of Bernstein's inequality on Euclidean space*

$$\|g\|_{\mathcal{H}^\alpha} \lesssim \max(b^{\alpha-\beta}, a^{\alpha-\beta})\|g\|_{\mathcal{H}^\beta}$$

and

$$\|g\|_{\mathcal{H}^\alpha} \gtrsim \min(b^{\alpha-\beta}, a^{\alpha-\beta})\|g\|_{\mathcal{H}^\beta}.$$

The former estimate still holds in the case where $a = 0$ and $\alpha > \beta$. For Littlewood-Paley projectors, this will be applied $b = 2a = 2^j$ for $j \in \mathbb{N}$.

Proof : The condition on g means that

$$g = \sum_{\lambda_k \in [a,b]} (g, \phi_k)\phi_k$$

and we have

$$\|g\|^2_{\mathcal{H}^\alpha} = \sum_{\lambda_k \in [a,b]} (g, \phi_k)^2 \lambda_k^{2\alpha}.$$

The upper bounds follow directly with

$$\lambda_k^{2\alpha} = \lambda_k^{2\beta} \lambda_k^{2(\alpha-\beta)} \leq \lambda^{2\beta} \max\left(b^{2(\alpha-\beta)}, a^{2(\alpha-\beta)}\right)$$

and analogously for the lower bounds.

□

The space $\mathcal{H}^\sigma$ is an algebra only for σ large enough depending on the dimension, this can be seen with the following Proposition and the Sobolev embedding. This type of estimates are important for the dispersive equations with cubic nonlinearity considered here.

Lemma 6.8. *Let $\sigma \geq 0$. The space $\mathcal{H}^\sigma \cap L^\infty$ is an algebra and one has the bound*

$$\|f \cdot g\|_{\mathcal{H}^\sigma} \lesssim \|f\|_{\mathcal{H}^\sigma}\|g\|_{L^\infty} + \|g\|_{\mathcal{H}^\sigma}\|f\|_{L^\infty}.$$

6.1.3 – Additional results on the Anderson Hamiltonian

In this Section, we provide two results on the Anderson Hamiltonian needed for our proof of Strichartz inequalities that follows directly from the construction of Chapter 4. Recall that for any $u \in \mathcal{D}_\Xi$, the operator H is given by

$$Hu = Lu^\sharp + \mathsf{P}_\xi u^\sharp + \mathsf{\Pi}(u^\sharp, \xi) + R(u)$$

with $u^\sharp = \Phi(u) \in \mathcal{H}^2$ and R an explicit operator depending on Ξ which is continuous from $\mathcal{H}^\alpha$ to $\mathcal{H}^{3\alpha-2}$. For each $s > 0$, we have a different representation of H, namely

$$Hu = H\Gamma u_s^\sharp = Lu_s^\sharp + \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R(\Gamma u_s^\sharp) + \Psi^s(\Gamma u_s^\sharp)$$

with $u_s^\sharp = \Phi^s(u) \in \mathcal{H}^2$ and Ψ^s an explicit operator depending on Ξ and s continuous from L^2 to C^∞ which we henceforth include in the operator R. The operator $H\Gamma$ is thus a perturbation of L, the following Proposition shows that it is a continuous operator from $\mathcal{H}^2$ to L^2. In Section 6.2.1, we show that it is even a lower order perturbation of the Laplace-Beltrami operator; this will be crucial to obtain Strichartz inequalities.

Proposition 6.9. *For any $\gamma \in (-\alpha, 3\alpha - 2)$ and s as above, we have*

$$\|H\Gamma u_s^\sharp\|_{\mathcal{H}^\gamma} \lesssim \|\Gamma u_s^\sharp\|_{\mathcal{H}^{\gamma+2}}.$$

In particular, the result holds for $\gamma \in (-1, 1)$ since the noise belongs to $\mathcal{C}^{\alpha-2}$ for any $\alpha < 1$.

Proof : We have

$$H\Gamma u_s^\sharp = Lu_s^\sharp + \mathsf{P}_\xi u_s^\sharp + \mathsf{\Pi}(u_s^\sharp, \xi) + R(u)$$

with $u = \Gamma u_s^\sharp$. Assume first that $0 < \gamma < 3\alpha - 2$ hence

$$\begin{aligned}\|H\Gamma u^\sharp\|_{\mathcal{H}^\gamma} &\lesssim \|Lu^\sharp\|_{\mathcal{H}^\gamma} + \|\mathsf{P}_\xi u^\sharp + \mathsf{\Pi}(u^\sharp, \xi)\|_{\mathcal{H}^\gamma} + \|R(u)\|_{\mathcal{H}^\gamma}\\ &\lesssim \|u^\sharp\|_{\mathcal{H}^{\gamma+2}} + \|\xi\|_{\mathcal{C}^{\alpha-2}}\|u^\sharp\|_{\mathcal{H}^{\gamma+2-\alpha}} + \|R(u)\|_{\mathcal{H}^{3\alpha-2}}\end{aligned}$$

where the condition $\gamma > 0$ is needed for the resonant term and $\gamma < 3\alpha - 2$ for $R(u)$. The result follows for this case since

$$\|R(u)\|_{\mathcal{H}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{H}^\alpha} \lesssim \|u^\sharp\|_{\mathcal{H}^\alpha} \lesssim \|u^\sharp\|_{\mathcal{H}^{\gamma+2}}.$$

Assume now that $-\alpha < \gamma \le 0$. For any $\delta > 0$, we have

$$\begin{aligned}\|H\Gamma u^\sharp\|_{\mathcal{H}^\gamma} &\lesssim \|Lu^\sharp\|_{\mathcal{H}^\gamma} + \|\mathsf{P}_\xi u^\sharp + \mathsf{\Pi}(u^\sharp, \xi)\|_{\mathcal{H}^\gamma} + \|R(u)\|_{\mathcal{H}^\gamma}\\ &\lesssim \|Lu^\sharp\|_{\mathcal{H}^\gamma} + \|\mathsf{P}_\xi u^\sharp + \mathsf{\Pi}(u^\sharp, \xi)\|_{\mathcal{H}^\delta} + \|R(u)\|_{\mathcal{H}^\gamma}\\ &\lesssim \|u^\sharp\|_{\mathcal{H}^{\gamma+2}} + \|\xi\|_{\mathcal{C}^{\alpha-2}}\|u^\sharp\|_{\mathcal{H}^{\delta+2-\alpha}} + \|R(u)\|_{\mathcal{H}^{3\alpha-2}}\end{aligned}$$

using that $\gamma \le 0 < \delta$. The proof is complete since $\gamma > -\alpha$ and δ small enough implies $\gamma + 2 > \delta + 2 - \alpha$.

□

While the regularity of a function can be measured by its coefficients in the basis of eigenfunction of the Laplacian, the same is true for the Anderson Hamiltonian and the spaces agree if the regularity one considers is below the form domain.

Proposition 6.10. *For $\beta \in (-\alpha, \alpha)$, there exists two constants $c_\Xi, C_\Xi > 0$ such that*

$$c_\Xi\|H^{\frac{\beta}{2}}u\|_{L^2} \le \|u\|_{\mathcal{H}^\beta} \le C_\Xi\|H^{\frac{\beta}{2}}u\|_{L^2}.$$

Proof : Observe first that the statement is clear for $\beta = 0$, we consider only the case $\beta \in (0, \alpha)$ since the case of negative β follows by duality. Again we take $(\varphi_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ to denote the basis of eigenfunctions of $-\Delta$ and H respectively. We have for any $v \in \mathcal{D}_\Xi$

$$
\begin{aligned}
\big\|H^{\frac{\beta}{2}} v\big\|_{L^2} &= \Big(\sum_{n\geq 1} \lambda_n^\beta \langle v, e_n\rangle^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{n\geq 1} \lambda_n^\beta \langle v, e_n\rangle^{2\beta} \langle v, e_n\rangle^{2-2\beta}\Big)^{\frac{1}{2}} \\
&\lesssim \big(\sum_{n\geq 1} \lambda_n \langle v, e_n\rangle^2\big)^{\frac{\beta}{2}} \big(\sum_{n\geq 1} \langle v, e_n\rangle^2\big)^{\frac{1-\beta}{2}} \\
&\lesssim \|H^{\frac{1}{2}} v\|_{L^2}^\beta \|v\|_{L^2}^{1-\beta}
\end{aligned}
$$

using Hölder's inequality. Thus the equivalence of $\|H^{\frac{1}{2}} v\|_{L^2}$ and $\|v_s^\sharp\|_{\mathcal{H}^1}$ together with the continuity of Φ^s from L^2 to itself yields

$$
\big\|H^{\frac{\beta}{2}} v\big\|_{L^2} \lesssim \|v_s^\sharp\|_{\mathcal{H}^1}^\beta \|v^\sharp\|_{L^2}^{1-\beta}.
$$

Applying this with $v = \Gamma\big(\langle u_s^\sharp, \varphi_n\rangle \varphi_n\big)$ gives

$$
\begin{aligned}
\big\|H^{\frac{\beta}{2}} \Gamma\big(\langle u^\sharp, \varphi_n\rangle \varphi_n\big)\big\|_{L^2} &\lesssim \|\langle u^\sharp, \varphi_n\rangle \varphi_n\|_{\mathcal{H}^1}^\beta \|\langle u^\sharp, \varphi_n\rangle \varphi_n\|_{L^2}^{1\beta} \\
&\lesssim |\langle u^\sharp, \varphi_n\rangle| \|\varphi_n\|_{\mathcal{H}^\beta}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|H^{\frac{\beta}{2}} u\|_{L^2}^2 = \|H^{\frac{\beta}{2}} \Gamma(u_s^\sharp)\|_{L^2}^2 &\leq \sum_{n\geq 1} \|H^{\frac{\beta}{2}} \Gamma\big(\langle u_s^\sharp, \varphi_n\rangle \varphi_n\big)\|_{L^2}^2 \\
&\lesssim \sum_{n\geq 1} |\langle u_s^\sharp, \varphi_n\rangle|^2 \|\varphi_n\|_{\mathcal{H}^\beta}^2 \\
&\lesssim \|u_s^\sharp\|_{\mathcal{H}^\beta}^2.
\end{aligned}
$$

Since $\beta \in [0, \alpha)$, we get

$$
\|H^{\frac{\beta}{2}} u\|_{L^2} \lesssim \|u\|_{\mathcal{H}^\beta}.
$$

from the boundedness of Γ. The other inequality follows from the same reasoning with

$$
\|v\|_{\mathcal{H}^\beta} \lesssim \|v_s^\sharp\|_{\mathcal{H}^\beta} \lesssim \|v_s^\sharp\|_{\mathcal{H}^1}^\beta \|v_s^\sharp\|_{L^2}^{1-\beta} \lesssim \|H^{\frac{1}{2}} v\|_{\mathcal{H}^1}^\beta \|u\|_{L^2}^{1-\beta}
$$

and applying this bound to $u = \sum_{n\geq 1} \langle u, e_n\rangle e_n$ and proceeding as above we get the other direction.

□

6.1.4 – Strichartz inequalities with white noise potential

As was hinted at in Proposition 5.5, the transformed operator

$$
H^\sharp := \Gamma^{-1} H \Gamma
$$

is a lower-order pertubation of the Laplace-Beltrami operator. This transformed operator appears naturally when transforming the Schrödinger equation and the wave equation with multiplicative noise, the continuity result on Γ and its inverse allow to relate results on $H^\sharp$ to H. We obtain the following result which is analogous to Proposition 3.2 in [59].

Proposition 6.11. *Let $0 \leq \beta < 1$. For any $\kappa > 0$, we have*

$$\|(H^\sharp - L)v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^{1+\beta+\kappa}}.$$

Proof : For $u = \Gamma u^\sharp \in \mathcal{D}_\Xi$, recall that

$$Hu = Lu^\sharp + \mathsf{P}_\xi u^\sharp + \Pi(u^\sharp, \xi) + R(u)$$

where

$$\begin{aligned} R(u) := {} & \Pi\big(u, \Pi(X_1, \xi)\big) + \mathsf{P}_{\Pi(X_1,\xi)} u + \mathsf{C}(u, X_1, \xi) + \mathsf{P}_u \Pi(X_2, \xi) + \mathsf{D}(u, X_2, \xi) \\ & + \mathsf{S}(u, X_2, \xi) + \mathsf{P}_\xi \widetilde{\mathsf{P}}_u X_2 - e^{-L}\left(\mathsf{P}_u X_1 + \mathsf{P}_u X_2\right). \end{aligned}$$

Thus $H^\sharp v$ is given by

$$H^\sharp v = Lv + \mathsf{P}_\xi v + \Pi(v, \xi) + R(\Gamma v) - \widetilde{\mathsf{P}}_{H\Gamma v}(X_1 + X_2)$$

and for any $\kappa > 0$ and $\beta \in [0, \alpha]$, we have

$$\begin{aligned} \|(H^\sharp - L)v\|_{\mathcal{H}^\beta} & \lesssim \|\mathsf{P}_\xi v + \Pi(v, \xi)\|_{\mathcal{H}^\beta} + \|R(\Gamma v)\|_{\mathcal{H}^\beta} + \|\widetilde{\mathsf{P}}_{H\Gamma v}(X_1 + X_2)\|_{\mathcal{H}^\beta} \\ & \lesssim \|\xi\|_{\mathcal{C}^{-1-\kappa}} \|v\|_{\mathcal{H}^{\beta+1+\kappa}} + \|\Gamma v\|_{\mathcal{H}^\alpha} + \|H\Gamma v\|_{\mathcal{H}^{-1+\kappa+\beta}} \|X_1 + X_2\|_{\mathcal{C}^{1-\kappa}} \\ & \lesssim \|v\|_{\mathcal{H}^{1+\beta+\kappa}} + \|v\|_{\mathcal{H}^\alpha} + \|v\|_{\mathcal{H}^{1+\kappa+\beta}} \end{aligned}$$

using Proposition 6.9 and the proof is complete since $\alpha < 1$.

$\square$

Since the unitary group associated to H is bounded on L^2 and on the domain $\mathcal{D}_\Xi$ of H, this implies a similar result for the "sharpened" group associated with $H^\sharp$ in terms of classical Sobolev spaces.

Proposition 6.12. *For any $0 \leq \beta \leq 2$ and $t \in \mathbb{R}$, we have*

$$\|e^{itH^\sharp} v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^\beta}.$$

Moreover, $e^{itH^\sharp}$ is a non-unitary group of L^2 bounded operators, namely

$$e^{i(t+s)H^\sharp} v = e^{itH^\sharp} e^{isH^\sharp} v$$

for all $s, t \in \mathbb{R}$ and $v \in L^2$.

Proof : For $\beta = 0$, this follows from the continuity of Γ and Γ^{-1} from L^2 to itself. For $\beta = 2$, Proposition 4.6 gives

$$(1 - \delta)\|\Gamma^{-1} u\|_{\mathcal{H}^2} \leq \|Hu\|_{L^2} + m_\delta^2(\Xi, s)\|u\|_{L^2}$$

thus

$$\begin{aligned} \|e^{itH^\sharp} v\|_{\mathcal{H}^2} & = \|\Gamma^{-1} e^{itH} \Gamma v\|_{\mathcal{H}^2} \\ & \lesssim \|H e^{itH} \Gamma v\|_{L^2} \\ & \lesssim \|e^{itH} H \Gamma v\|_{L^2} \\ & \lesssim \|H \Gamma v\|_{L^2} \\ & \lesssim \|v\|_{\mathcal{H}^2}. \end{aligned}$$

The results for any $\beta \in (0, 2)$ is obtained by interpolation and the group property follows simply from the group property of e^{itH} by observing

$$e^{i(t+s)H^\sharp} v = \Gamma^{-1} e^{i(t+s)H} \Gamma v = \Gamma^{-1} e^{itH} \Gamma \Gamma^{-1} e^{isH} \Gamma v = e^{itH^\sharp} e^{isH^\sharp v}$$

□

Strichartz inequalities are refinement of the estimates from the previous Proposition. The following statement is such a result, which has an arbitraty small loss of derivative coming from the irregularity of the noise in the addition to the $\frac{1}{p}$ loss from the manifold setting which one sees in [19]. We refer to a pair (p,q) statisfying

$$\frac{2}{p}+\frac{2}{q}=1$$

as a Strichartz pair from here.

Theorem 6.13. *Let (p,q) be a Strichartz pair. Then for any $\kappa>0$*

$$\|e^{itH^\sharp}v\|_{L^p([0,1],L^q)}\lesssim\|v\|_{\mathcal{H}^{\frac{1}{p}+\kappa}}.$$

First, we need to prove the following Lemma. It gives the difference between the Schrödinger groups associated to $H^\sharp$ and L from the difference between $H^\sharp$ and L itself. Moreover it quantifies that their difference is small in a short time interval if one gives up some regularity.

Lemma 6.14. *Given $v\in\mathcal{H}^2$, we have*

$$\Big(e^{i(t-t_0)H^\sharp}-e^{i(t-t_0)L}\Big)v=i\int_{t_0}^t e^{i(t-s)L}(H^\sharp-L)e^{i(s-t_0)H^\sharp}v\mathrm{d}s$$

for any $t,t_0\in\mathbb{R}$.

Proof : The "sharpened" group yields the solution of the Schrödinger equation

$$\big(i\partial_t+H^\sharp\big)(e^{itH^\sharp}v)=0$$

thus

$$\big(i\partial_t+L\big)(e^{itH^\sharp}v)=\big(L-H^\sharp\big)(e^{itH^\sharp}v).$$

Using the unitary group representation of the solution to the Schrödinger equation associated to L, we deduce that

$$\big(i\partial_t+L\big)(e^{itL}v-e^{itH^\sharp}v)=\big(H^\sharp-L\big)(e^{itH^\sharp}v).$$

Since the solution is equal to 0 at time 0, the mild formulation of this last equation yields

$$\Big(e^{itH^\sharp}-e^{itL}\Big)v=i\int_0^t e^{i(t-s)L}(H^\sharp-L)e^{isH^\sharp}v\mathrm{d}s.$$

The result for any $t_0\in\mathbb{R}$ follows from the same proof.

□

Proof of Theorem 6.13 : For $N\in\mathbb{N}^*$ to be chosen, we have

$$\|e^{itH^\sharp}v\|^p_{L^p([0,1],L^q)}=\sum_{\ell=0}^N\|e^{itH^\sharp}v\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)}$$

where $t_\ell := \frac{\ell}{N}$. For $t \in [t_\ell, t_{\ell+1})$, the previous Lemma gives

$$e^{itH^\sharp}v = e^{i(t-t_\ell)H^\sharp}e^{it_\ell H^\sharp}v = e^{i(t-t_\ell)L}e^{it_\ell H^\sharp}v + i\int_{t_\ell}^t e^{i(t-s)L}(H^\sharp - L)e^{isH^\sharp}v\mathrm{d}s.$$

Applying this with $v = \Delta_k u$ gives

$$\begin{aligned}\|\Delta_j e^{itH^\sharp}\Delta_k u\|^p_{L^p([0,1],L^q)} \leq &\sum_{\ell=0}^N \|\Delta_j e^{i(t-t_\ell)L}e^{it_\ell H^\sharp}\Delta_k u\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)} \\ &+ \sum_{\ell=0}^N \left\|\Delta_j\int_{t_\ell}^t e^{i(t-s)L}(H^\sharp - L)e^{isH^\sharp}\Delta_k u\mathrm{d}s\right\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)}.\end{aligned}$$

Assume $N \geq 2^j$ such that $|t_{\ell+1} - t_\ell| \leq 2^{-j}$. For the first term, we have

$$\begin{aligned}\|\Delta_j e^{i(t-t_\ell)L}e^{it_\ell H^\sharp}\Delta_k u\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)} &= \|e^{i(t-t_\ell)L}\Delta_j e^{it_\ell H^\sharp}\Delta_k u\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)} \\ &\lesssim \|\Delta_j e^{it_\ell H^\sharp}\Delta_k u\|^p_{L^2} \\ &\lesssim 2^{-j\delta p}\|\Delta_j e^{it_\ell H^\sharp}\Delta_k u\|^p_{\mathcal{H}^\delta} \\ &\lesssim 2^{-j\delta p}2^{-k\delta' p}\|\Delta_k u\|^p_{\mathcal{H}^{\delta+\delta'}}\end{aligned}$$

for any $\delta, \delta' \in \mathbb{R}$ using Proposition 6.12, Strichartz inequality for spectrally localised data from Section 6.1.2 and Bernstein's Lemma, see Lemma 6.7. For the second term, we have

$$\begin{aligned}&\left\|\Delta_j\int_{t_\ell}^t e^{i(t-s)L}(H^\sharp - L)e^{isH^\sharp}\Delta_k u\mathrm{d}s\right\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)} \\ &\qquad= \left\|\int_{t_\ell}^t e^{i(t-s)L}\Delta_j(H^\sharp - L)e^{isH^\sharp}\Delta_k u\mathrm{d}s\right\|^p_{L^p([t_\ell,t_{\ell+1}],L^q)} \\ &\qquad\lesssim \left(\int_{t_\ell}^{t_{\ell+1}} \|e^{i(t-s)L}\Delta_j(H^\sharp - L)e^{isH^\sharp}\Delta_k u\|_{L^p([t_\ell,t_{\ell+1}],L^q)}\mathrm{d}s\right)^p \\ &\qquad\lesssim \left(\int_{t_\ell}^{t_{\ell+1}} \|\Delta_j(H^\sharp - L)e^{isH^\sharp}\Delta_k u\|_{L^2}\mathrm{d}s\right)^p \\ &\qquad\lesssim 2^{-j\sigma p}\left(\int_{t_\ell}^{t_{\ell+1}} \|\Delta_j(H^\sharp - L)e^{isH^\sharp}\Delta_k u\|_{\mathcal{H}^\sigma}\mathrm{d}s\right)^p \\ &\qquad\lesssim 2^{-j\sigma p}\left(\int_{t_\ell}^{t_{\ell+1}} \|(H^\sharp - L)e^{isH^\sharp}\Delta_k u\|_{\mathcal{H}^\sigma}\mathrm{d}s\right)^p \\ &\qquad\lesssim 2^{-j\sigma p}\left(\int_{t_\ell}^{t_{\ell+1}} \|e^{isH^\sharp}\Delta_k u\|_{\mathcal{H}^{\sigma+1+\kappa}}\mathrm{d}s\right)^p \\ &\qquad\lesssim N^{-p}2^{-j\sigma p}\|\Delta_k u\|^p_{\mathcal{H}^{1+\sigma+\kappa}} \\ &\qquad\lesssim N^{-p}2^{-j\sigma p}2^{-k\sigma' p}\|\Delta_k u\|^p_{\mathcal{H}^{1+\sigma+\sigma'+\kappa}}\end{aligned}$$

for any $\sigma \in (0,1), \sigma' \in \mathbb{R}$ and $0 < \kappa < 1 - \alpha$ where again the dyadic factors come from Bernstein's Lemma and we have used the bounds from Propositions 6.11 and 6.12 with Strichartz inequality for spectrally localised data. Summing over the sub-intervals gives

$$\|\Delta_j e^{itH^\sharp}\Delta_k u\|_{L^p([0,1],L^q)} \lesssim N^{\frac{1}{p}}2^{-j\delta}2^{-k\delta'}\|\Delta_k u\|_{\mathcal{H}^{\delta+\delta'}} + N^{\frac{1-p}{p}}2^{-j\sigma}2^{-k\sigma'}\|\Delta_k u\|_{\mathcal{H}^{1+\sigma+\sigma'+\kappa}}.$$

Let $\eta > 0$ small and take

$$N = 2^j,\ \delta = \eta + \frac{1}{p},\ \delta' = \sigma = \sigma' = \eta$$

which satisfies in particular $N \geq 2^j$ and $\sigma \in [0, \alpha)$ to sum over $k \leq j$. We get

$$\begin{aligned}
\|\sum_{k\leq j} \Delta_j e^{itH^\sharp}\Delta_k u\|_{L^p([0,1],L^q)} &\lesssim \sum_j \sum_{k\leq j} \|\Delta_j e^{itH^\sharp}\Delta_k u\|_{L^p([0,1],L^q)} \\
&\lesssim \sum_{j\geq 0}\sum_{k\leq j} 2^{\frac{j}{p}} 2^{-j(\frac{1}{p}+\eta)} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{\frac{1}{p}+2\eta}} + 2^{j\frac{1-p}{p}} 2^{-j\eta} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{1+2\eta+\kappa}} \\
&\lesssim \sum_{j\geq 0} 2^{-j\eta} \|\Delta_{\leq j} u\|_{\mathcal{H}^{\frac{1}{p}+2\eta}} + 2^{j\frac{1-p}{p}} 2^{-j\eta} \|\Delta_{\leq j} u\|_{\mathcal{H}^{1+2\eta+\kappa}} \\
&\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}+2\eta}} + \sum_{j\geq 0} 2^{-j\eta} 2^{j\frac{1-p}{p}} 2^{-j\frac{1-p}{p}} \|\Delta_{\leq j} u\|_{\mathcal{H}^{1-1+\frac{1}{p}+2\eta+\kappa}} \\
&\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}+2\eta}} + \|u\|_{\mathcal{H}^{\frac{1}{p}+2\eta+\kappa}},
\end{aligned}$$

having used Bernstein's inequality, Lemma 6.7, for the projector $\Delta_{\leq j}$. For the sum over $j \leq k$, we choose instead

$$N = 2^k,\ \delta = \delta' = \sigma = \sigma' = \eta,$$

with $\eta > 0$ small as before. Since $j \leq k$, we have $N \geq 2^j$ thus get the bound for the other part of the double sum

$$\begin{aligned}
\|\sum_{j\leq k} \Delta_j e^{itH^\sharp}\Delta_k u\|_{L^p([0,1],L^q)} &\lesssim \sum_{k\geq 0} \sum_{j\leq k} \|\Delta_j e^{itH^\sharp}\Delta_k u\|_{L^p([0,1],L^q)} \\
&\lesssim \sum_{k\geq 0}\sum_{j\leq k} 2^{\frac{k}{p}} 2^{-j\eta} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{2\eta}} + 2^{\frac{k(1-p)}{p}} 2^{-j\eta} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{1+2\eta+\kappa}} \\
&\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}+2\eta}} + \|u\|_{\mathcal{H}^{1+\frac{1-p}{p}+2\eta+\kappa}} \\
&\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}+2\eta+\kappa}},
\end{aligned}$$

having used Bernstein's inequality again. This completes the proof since η and κ can be taken arbitrary small.

□

Remark : *We proved that Strichartz inequalities are stable under suitable perturbation, that is lower-order perturbation in the sense of previous Proposition 6.11. One can show that the magnetic Laplacian with white noise magnetic field constructed in [49] is also a lower order perturbation of the Laplacian on the two-dimensional torus in this sense. Thus Theorem 6.13 also gives Strichartz inequalities for the associated Schrödinger group. Also, the different continuity results for* Γ *and its inverse impliy the same bounds for the group associated to* H *and not* $H^\sharp$.

As Corollary, we state the inhomogeneous inequalities needed to solve the nonlinear equation. This is straightforward and we ommit the proof, see [59] and references therein.

Corollary 6.15. *In the setting of Theorem 6.13, we have in addition the bound*

$$\left\|\int_0^t e^{-i(t-s)H^\sharp} f(s)\mathrm{d}s\right\|_{L^p([0,1],L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\frac{1}{p}+\varepsilon}}\mathrm{d}s$$

for all $f \in L^1([0,1],\mathcal{H}^{\frac{1}{p}+\kappa})$.

The only ingredient in the proof of the Theorem where the boundary appears is when we apply the result for the Laplacian. Theorem 6.6 immediatly gives the following result.

Theorem 6.16. *Let $p \in (2,\infty]$ and $q \in [3,\infty)$ such that*

$$\frac{3}{p}+\frac{2}{q}=1.$$

Then for any $\kappa > 0$

$$\|e^{-itH^\sharp}u\|_{L^p([0,1],L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{2}{p}+\kappa}}.$$

and

$$\left\|\int_0^t e^{-i(t-s)H^\sharp} f(s)\mathrm{d}s\right\|_{L^p([0,1],L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\frac{2}{p}+\kappa}}\mathrm{d}s$$

for all $f \in L^1([0,1],\mathcal{H}^{\frac{2}{p}+\kappa})$.

6.1.5 – Local well-posedness in low-regularity Sobolev spaces

We now apply our results to the local in time well-posedness of the cubic multiplicative stochastic NLS

$$\left|\begin{array}{rcl} i\partial_t u - Hu &=& -|u|^2u \\ u(0) &=& u_0 \end{array}\right.$$

with $u_0 \in \mathcal{H}^\sigma$ where $\sigma \in (\frac{1}{2},1)$ and in the energy space, that is $u_0 \in \mathcal{D}(\sqrt{H}) = \Gamma\mathcal{H}^1$ the form domain. As explained in Section 3.2.2 of [38], their result for the equation with white noise potential is weaker than the one for the deterministic equation since Strichartz inequalities were not know in this singular case. This was a motivation for the work [59] while our result allows to solve the equation in a regime arbitrary closed to the one of the deterministic equation on a surface. This Section follows the line of this last paper, we recall the core ideas nonetheless. Finally, we only consider a surface without boundary, the case with boundary is analogue with the associated Strichartz inequalities. The mild formulation is

$$u(t) = e^{-itH}u_0 + i\int_0^t e^{-i(t-s)H}\big(|u|^2u\big)(s)\mathrm{d}s$$

and applying the Γ^{-1} map introduced in Chapter 4 yields the mild formulation for the transformed quantity $u^\sharp = \Gamma^{-1}u$. We get

$$u^\sharp(t) = e^{-itH^\sharp}u_0^\sharp + i\int_0^t e^{-i(t-s)H^\sharp}\Gamma^{-1}\big(|\Gamma u^\sharp|^2\Gamma u^\sharp\big)(s)\mathrm{d}s$$

where $u_0^\sharp := \Gamma^{-1}u_0$, this is where the transformed operator $H^\sharp = \Gamma^{-1}H\Gamma$ appears naturally. Despite the seemingly complicated nonlinear expression, this new mild formulation is easier to deal with since $H^\sharp$ is a perturbation of the Laplacian and has domain $\mathcal{H}^2$, hence it is not as outlandish as H and its domain which contains no non-zero smooth functions. Now, we have to find a bound for the map

$$\Psi(v)(t) := e^{-itH^\sharp}v_0 + i\int_0^t e^{-i(t-s)H^\sharp}\Gamma^{-1}\big(|\Gamma v|^2\Gamma v\big)(s)\mathrm{d}s$$

in a suitable space which allows us to get a unique fixed point. One then recovers a solution to the original equation with $u := \Gamma v$ and choosing $v_0 := \Gamma^{-1}u_0$. Since both $e^{-itH^\sharp}$ and Γ are bicontinuous from L^p to itself for $p \in [2,\infty]$ and from $\mathcal{H}^\sigma$ to itself for $\sigma \in [0,1)$, it is natural to consider initial datum v_0, and thus also u_0, in $\mathcal{H}^\sigma$ for $0 < \sigma < 1$. Therefore we bound $\Psi(v)$ in $\mathcal{H}^\sigma$ with

$$\begin{aligned}
\big\|\Psi(v)(t)\big\|_{\mathcal{H}^\sigma} &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|\Gamma v(s)^3\|_{\mathcal{H}^\sigma}\mathrm{d}s\\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|\Gamma v(s)\|_{\mathcal{H}^\sigma}\|\Gamma v(s)\|_{L^\infty}^2\mathrm{d}s\\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|v(s)\|_{\mathcal{H}^\sigma}\|v(s)\|_{L^\infty}^2\mathrm{d}s\\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|_{L^2([0,t],L^\infty)}^2
\end{aligned}$$

where in the first and third lines we have used the continuity of $e^{-itH^\sharp}$ and Γ and Bernstein Lemma 6.8 in the second line. For $\sigma < 1$, the space $\mathcal{H}^\sigma$ is not an algebra and one can not simply use its norm to bound the nonlinearity. However, one may bound it using the L^∞-norm in space by observing that one needs less integrability in time and this is precisely the point where the Strichartz estimates turn out to be useful. As for the deterministic equation, we work with the function spaces

$$\mathcal{W}^{\beta,q}(M) = \{u \in \mathcal{D}'(M); (1-\Delta)^{\frac{\beta}{2}}u \in L^q\}$$

with associated norm

$$\|u\|_{\mathcal{W}^{\beta,q}} := \|(1-\Delta)^{\frac{\beta}{2}}u\|_{L^q}.$$

For $\beta \in [0,1)$ and $q = 2$, one recovers the Sobolev spaces and the norm is equivalent to

$$\|u\|_{\mathcal{H}^\beta} = \|(1+H)^{\frac{\beta}{2}}u\|_{L^2}$$

by Proposition 6.10. Within this framework, Strichartz inequalities from Theorem 6.13 gives us the bound

$$\|e^{-itH^\sharp}w\|_{L^p([0,1],\mathcal{W}^{\beta,q})} \lesssim \|w\|_{\mathcal{H}^{\frac{1}{p}+\beta+\kappa}},$$

for any Strichartz pair (p,q) and $\kappa > 0$. Furthermore, the space $\mathcal{W}^{\beta,q}$ is continuously embedded in L^∞ for $\beta \geq \frac{2}{q}$. Let $\sigma \in \mathbb{R}$ such that

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma.$$

Thus for $0 < t \leq 1$, we get the bound

$$\begin{aligned}
\|\Psi(v)\|_{L^p\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa,q}\right)} &\lesssim \|v_0\|_{\mathcal{H}^{\frac{1}{p}+\frac{2}{q}+2\kappa}} + \int_0^t \left\|\Gamma^{-1}\big(|\Gamma v|^2\Gamma v\big)(s)\right\|_{\mathcal{H}^{\frac{1}{p}+\frac{2}{q}+2\kappa}}\mathrm{d}s \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \left\|\Gamma v(s)^3\right\|_{\mathcal{H}^\sigma}\mathrm{d}s \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^2([0,t],L^\infty)} \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^2\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa}\right)} \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + t^{\frac{p-2}{p}}\|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^p\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa}\right)}
\end{aligned}$$

using Corollary 6.15 in the first line, Hölder inequality in the last line and bicontinuity of Γ from $\mathcal{H}^\sigma$ to itself. For $0 < t' \leq t$, we also have

$$\begin{aligned}
\left\|\Psi(v)(t')\right\|_{\mathcal{H}^\sigma} &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^{t'} \|v(s)\|_{\mathcal{H}^\sigma}\|v(s)\|^2_{L^\infty}\mathrm{d}s \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^2([0,t],L^\infty)}, \\
&\lesssim \|v_0\|_{\mathcal{H}^\sigma} + t^{\frac{p-2}{p}}\|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^p\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa}\right)}.
\end{aligned}$$

This gives us the combined bound

$$\|\Psi(v)\|_{L^p\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa,q}\right)} + \|\Psi(v)\|_{L^\infty([0,t],\mathcal{H}^\sigma)} \lesssim \|v_0\|_{\mathcal{H}^\sigma} + t^{\frac{p-2}{p}}\|v\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|v\|^2_{L^p\left([0,t],\mathcal{W}^{\frac{2}{q}+\kappa}\right)}$$

that will be the main tool for the fixed point. Remark that the restrictions

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma \quad \text{and} \quad \frac{2}{p} + \frac{2}{q} = 1$$

gives

$$1 - \frac{1}{p} + 2\kappa \leq \sigma.$$

Since $p \geq 2$ and $\kappa > 0$ can be taken arbitrary small, this gives

$$\sigma > \frac{1}{2}$$

and leads to the following local-in-time well-posedness result. Without Strichartz estimates, one could not go beyond the limit $\sigma \geq 1$.

Theorem 6.17. *Let $\sigma > \frac{1}{2}$ and initial data $v_0 \in \mathcal{H}^\sigma$. Let $\kappa > 0$ and (p, q) a Strichartz pair such that*

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma,$$

there exists a time $T > 0$ until which there exists a unique solution

$$v \in C\big([0,T],\mathcal{H}^\sigma\big) \cap L^p\big([0,T],\mathcal{W}^{\frac{2}{q}+\kappa}\big)$$

to the mild formulation of the transformed PDE

$$\left|\begin{array}{rcl} i\partial_t v - H^\sharp v &=& -\Gamma^{-1}\big(|\Gamma v|^2\Gamma v\big) \\ v(0) &=& v_0 \end{array}\right. .$$

Moreover, the solution depends continuously on the initial data $v_0 \in \mathcal{H}^\sigma$.

Proof : This is a straightforward contraction argument where the main ingredient is the bound proved in the preceding arguments. By choosing the radius R of the ball and the final time appropriately, we can prove that

$$\Psi : B(0,R)_{C([0,T],\mathcal{H}^\sigma)\cap L^p(([0,T],\mathcal{W}^{\frac{2}{q}+\kappa})} \to B(0,R)_{C([0,T],\mathcal{H}^\sigma)\cap L^p(([0,T],\mathcal{W}^{\frac{2}{q}+\kappa})}$$

is in fact a contraction, we refer to [59] for the details.

□

Finally we give the analogous result for surfaces with boundary which is of course weaker, however we still get a better result than one gets simply from using the algebra property of Sobolev spaces.

Theorem 6.18. *Let M be a compact surface with boundary, $\sigma > \frac{2}{3}$ and p, q, κ s.t.*

$$\frac{3}{p} + \frac{2}{q} = 1 \text{ and } \frac{2}{p} + \frac{2}{q} + 2\kappa \leq \sigma.$$

For any initial datum $v_0 \in \mathcal{H}^\sigma$ there exists a unique solution

$$v \in C([0,T],\mathcal{H}^\sigma) \cap L^p([0,T], W^{\frac{2}{q}+\kappa})$$

to the mild formulation of the transformed PDE up to a time $T > 0$ depending on the data which depends continuously on the initial condition.

6.2 – Nonlinear wave equation

Again, we consider the “sharpened”operator

$$H^\sharp := \Gamma^{-1} H \Gamma$$

which appears naturally when transforming the wave equation with multiplicative noise. If u solves

$$\left| \begin{array}{rcl} \partial_t^2 u + Hu &=& 0 \\ (u, \partial_t u)|_{t=0} &=& (u_0, u_1) \end{array} \right.$$

then $u^\sharp := \Gamma^{-1} u$ solves the transformed equation

$$\left| \begin{array}{rcl} \partial_t^2 u^\sharp + H^\sharp u^\sharp &=& 0 \\ (u^\sharp, \partial_t u^\sharp)|_{t=0} &=& (\Gamma^{-1}u_0, \Gamma^{-1}u_1) \end{array} \right.$$

In this Section, we show Strichartz inequalities for the associated wave equation. In the particular case of the torus, the improved Strichartz estimates for the Schrödinger equations yields better bounds on the eigenfunctions of the Anderson Hamiltonian thus improved Strichartz estimates for the wave equation. Afterwards, we detail how these can be used to get a low-regularity solution theory for the nonlinear wave equation with multiplicative noise. The propagator associated to the wave equation is

$$(u_0, u_1) \mapsto \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} u_1$$

with initial conditons $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. As for the Schrödinger equation, we first recall the deterministic result for the deterministic PDE. The following

Strichartz inequalities hold on a two-dimensional compact Riemannian manifold without boundary, see [12] and the references therein. We state the result in the homogeneous case for simplicity however one has similar results for the inhomogenous case, see Corollary 6.15. We cite the following Strichartz estimates which hold on compact surfaces respectively without and with boundary, see [12].

Theorem 6.19. *Let (M, g) be a compact two-dimensional Riemannian manifold without boundary. Let $p, q \in [2, \infty]$ such that*

$$\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}$$

and consider

$$\frac{1}{p} + \frac{2}{q} := 1 - \sigma.$$

Then the solution to

$$\begin{aligned} (\partial_t^2 - \Delta_g)u &= 0 \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1} \end{aligned}$$

satisfies the bound

$$\|u\|_{L^p([0,T],L^q)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}}.$$

In the case where the surface M has a boundary, there is this slightly weaker result.

Theorem 6.20. *Let (M, g) be a compact two-dimensional Riemannian manifold with boundary. Let $p \in (2, \infty]$ and $q \in [2, \infty)$ such that*

$$\frac{3}{p} + \frac{1}{q} \leq \frac{1}{2}$$

and consider σ given by

$$\frac{1}{p} + \frac{2}{q} = 1 - \sigma.$$

Then the solution to

$$\begin{aligned} (\partial_t^2 - \Delta_g)u &= 0 \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1} \end{aligned}$$

satisfies the bound

$$\|u\|_{L^p([0,T],L^q)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}}.$$

6.2.1 – Strichartz inequalities for the wave equation with multiplicative noise

While our proof of the Strichartz inequalities for the Schrödinger equation with white noise potential strongly relies on the deterministic result, this is not the case for the wave equation. In this case, we follow the approach from [20] for which one has two mains ingredients, firstly a Weyl law for the Laplace-Beltrami operator and secondly L^q bounds on its eigenfunctions. In particular, we treat at the same time the case with and without boundary here. An analogous Weyl law for the

Anderson Hamiltonian was obtained in Chapter 4 and the analogue of the second part follows from the Strichartz inequalities for the Schrödinger group obtained in Section 6.1.4. Let $(e_n)_{n\geq 1}$ be an orthonormal family of eigenfunctions of H associated to $\big(\lambda_n(\Xi)\big)_{n\geq 1}$. Since the eigenfuctions belong to the domain $\mathcal{D}_\Xi$, they belong in particular to L^∞ and we have the following bounds on its L^q-norm for $q \in (2,\infty)$. As stated in the remark after the proof of Theorem 6.13, the bounds are valid for the group associated to H as well as $H^\sharp$. In particular, the following bounds rely on the Strichartz inequalities for the Schrödinger group associated to H.

Proposition 6.21. *Let $q \in (2,\infty)$. We have*

$$\|e_n\|_{L^q} \lesssim \sqrt{\lambda_n(\Xi)}^{\frac{1}{2}-\frac{1}{q}+\kappa}$$

for any $\kappa > 0$. In particular, this implies

$$\|e_n\|_{L^q} \lesssim (1+\sqrt{n})^{\frac{1}{2}-\frac{1}{q}+\kappa}.$$

Proof : We have

$$\|e_n\|_{L^q} = \|e^{it\lambda_n} e_n\|_{L^p([0,1],L^q)} = \|e^{itH} e_n\|_{L^p([0,1],L^q)}$$

with (p,q) a Strichartz pair. For any $\kappa > 0$, this gives

$$\begin{aligned}
\|e_n\|_{L^q} &\lesssim \|e_n\|_{\mathcal{H}^{\frac{1}{p}+\kappa}} \\
&\lesssim \|\sqrt{H}^{\frac{1}{p}+\kappa} e_n\|_{L^2} \\
&\lesssim \sqrt{\lambda_n(\Xi)}^{\frac{1}{p}+\kappa}
\end{aligned}$$

using Proposition 6.10 and

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}.$$

Using the bound on the eigenvalues of the Anderson Hamiltonian

$$\lambda_n(\Xi) \lesssim 1 + \lambda_n(0),$$

the proof is complete using the asymptotics of the eingenvalues of the Laplace-Beltrami operator.

□

Another important operator is the projection onto the eigenspaces of H. Let

$$\Pi_\lambda u := \sum_{\lambda_n(\Xi)\in[\lambda,\lambda+1)} \langle u, e_n\rangle e_n$$

for any $\lambda \geq 0$. The Weyl law for H together with the bounds on e_n gives the following bound on Π_λ.

Proposition 6.22. *Let $\lambda \geq 0$ and $q \in (2,\infty)$. We have*

$$\|\Pi_\lambda u\|_{L^q} \lesssim \sqrt{\lambda+1}^{\frac{1}{2}-\frac{1}{q}+\kappa} \|u\|_{L^2}$$

for any $\kappa > 0$.

Proof : The result follows from the direct computation

$$\begin{aligned}\|\Pi_\lambda u\|_{L^q} &= \sum_{\lambda_n(\Xi)\in[\lambda,\lambda+1)} \langle u, e_n\rangle \|e_n\|_{L^q} \\ &\lesssim \sum_{\lambda_n(\Xi)\in[\lambda,\lambda+1)} \|u\|_{L^2}\|e_n\|_{L^2}\|e_n\|_{L^q} \\ &\lesssim \|u\|_{L^2}\big|\{\lambda_n(\Xi)\in[\lambda,\lambda+1)\}\big|\sqrt{\lambda+1}^{\frac12-\frac1q+\kappa} \\ &\lesssim \|u\|_{L^2}\sqrt{\lambda+1}^{\frac12-\frac1q+\kappa}\end{aligned}$$

for any $\kappa > 0$ since the almost sure Weyl-type law implies

$$\sup_{\lambda>0} |\{\lambda_n(\Xi)\in[\lambda,\lambda+1)\}| < \infty.$$

□

As mentioned before, this is the point where there are slightly weaker results in the case of a surface with boundary. We use Theorem 6.16 instead.

Proposition 6.23. *Let $q \in (2,\infty)$ and M a compact surface with boundary. We have*

$$\|e_n\|_{L^q} \lesssim \sqrt{\lambda_n(\Xi)}^{\frac23-\frac{4}{3q}+\kappa}$$

for any $\kappa > 0$. In particular, this implies

$$\|e_n\|_{L^q} \lesssim (1+\sqrt{n})^{\frac23-\frac{4}{3q}+\kappa}.$$

Moreover, for the operator Π_λ we have

$$\|\Pi_\lambda u\|_{L^q} \lesssim \sqrt{\lambda+1}^{\frac23-\frac{4}{3q}+\kappa}\|u\|_{L^2}$$

for any $\kappa > 0$.

Let B be the operator defined by

$$Be_n := \lfloor\sqrt{\lambda_n(\Xi)}\rfloor e_n$$

for any $n \geq 1$. The following Proposition gives continuity estimates for the unitary groups associated to $\sqrt{H}$ and B and bound the difference between the two operators.

Proposition 6.24. *For any $\beta \in [0,1)$ and $t \in \mathbb{R}$, we have*

$$\|e^{it\sqrt{H}}u\|_{\mathcal{H}^\beta} \lesssim \|u\|_{\mathcal{H}^\beta}$$

and

$$\|e^{itB}u\|_{\mathcal{H}^\beta} \lesssim \|u\|_{\mathcal{H}^\beta}.$$

Moreover, the difference $B-\sqrt{H}$ is bounded on $\mathcal{H}^\beta$ for any $\beta \in [0,1)$ and the difference between the semigroups is given by

$$e^{itB}u - e^{it\sqrt{H}} = -i\int_0^t e^{i(t-s)B}(\sqrt{H}-B)e^{is\sqrt{H}}\mathrm{d}s.$$

Proof : We have
$$\|e^{it\sqrt{H}}v\|_{L^2} \lesssim \|v\|_{L^2}.$$
thus
$$\|H^{\frac{\beta}{2}}e^{it\sqrt{H}}v\|_{L^2} = \|e^{it\sqrt{H}}H^{\frac{\beta}{2}}v\|_{L^2} \lesssim \|H^{\frac{\beta}{2}}v\|_{L^2}$$
for any $\beta \in (0,\alpha)$. Using Proposition 6.10, this gives
$$\|e^{it\sqrt{H}}v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^\beta}$$
and the result for e^{itB} follows from this. For the difference, $\|B-\sqrt{H}\|_{L^2\to L^2}$ is bounded by 1 and we have
$$\begin{aligned}\|H^{\frac{\beta}{2}}(B-\sqrt{H})u\|_{L^2} &= \|(B-\sqrt{H})H^{\frac{\beta}{2}}u\|_{L^2}\\ &\leq \|H^{\frac{\beta}{2}}u\|_{L^2}\end{aligned}$$
hence the boundeness of $B-\sqrt{H}$ on $\mathcal{H}^\beta$. The result on the difference of semigroup
$$e^{itB}u - e^{it\sqrt{H}} = -i\int_0^t e^{i(t-s)B}(\sqrt{H}-B)e^{is\sqrt{H}}\mathrm{d}s$$
follows with the same reasoning as in Lemma 6.14.

$\square$

We now have all the ingredients to prove of the Strichartz inequalities for the wave propagator associated to the Anderson Hamiltonian.

Theorem 6.25. *Let M be a compact surface without boundary $(p,q)\in[2,\infty)^2$ and $0<\sigma<\alpha$ such that $p\leq q$ and*
$$\sigma = \frac{3}{2}-\frac{2}{p}+\frac{1}{q}.$$
Then for any $\kappa>0$, we have the bound
$$\Big\|\cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1\Big\|_{L^p([0,1],L^q)} \lesssim \|(u_0,u_1)\|_{\mathcal{H}^{\sigma+\kappa}\times\mathcal{H}^{\sigma-1+\kappa}}.$$

Proof : We start by proving the bound for e^{itB} using the spectral decomposition
$$e^{itB}u = \sum_{n\geq 1} e^{itn}\Pi_n u$$
and then bound the difference of the two groups. First, the condition $p\leq q$ implies
$$\|e^{itB}u\|_{L^p([0,1],L^q(M))} \leq \|e^{itB}u\|_{L^q(M,L^p([0,1]))}$$
hence it is enough to bound the right hand side. Using the Sobolev embedding in the time variable and the L^q bound on the eigenvalues from Proposition 6.21, we

have

$$
\begin{aligned}
\|e^{itB}u\|^2_{L^q(M,L^p([0,1]))} &= \Big\| \|e^{itB}u\|^2_{L^p([0,1])} \Big\|_{L^{\frac{q}{2}}(M)} \\
&\lesssim \Big\| \|e^{itB}u\|^2_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])} \Big\|_{L^{\frac{q}{2}}(M)} \\
&\lesssim \sum_{n\geq 0} \Big\| \|e^{itn}\Pi_n u\|^2_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])} \Big\|_{L^{\frac{q}{2}}(M)} \\
&\lesssim \sum_{n\geq 0} \|e^{itn}\|^2_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])} \|\Pi_n u\|^2_{L^q(M)} \\
&\lesssim \sum_{n\geq 0} (n+1)^{1-\frac{2}{p}} (\sqrt{n}+1)^{1-\frac{2}{q}+2\kappa} \|\Pi_n u\|^2_{L^2} \\
&\lesssim \|\sqrt{H}^{\frac{3}{2}-\frac{2}{p}-\frac{1}{q}+\kappa} u\|^2_{L^2} \\
&\lesssim \|u\|^2_{\mathcal{H}^{\frac{3}{2}-\frac{2}{p}-\frac{1}{q}+\kappa}}
\end{aligned}
$$

which gives the result for B. To obtain the proof for $\sqrt{H}$, we use

$$
e^{itB}u - e^{it\sqrt{H}} = -i \int_0^t e^{i(t-s)B}(\sqrt{H}-B)e^{is\sqrt{H}} \mathrm{d}s.
$$

Indeed, this gives

$$
\begin{aligned}
\|e^{it\sqrt{H}}u\|_{L^p([0,1],L^q)} &\lesssim \|e^{itB}u\|_{L^p([0,1],L^q)} + \int_0^1 \|e^{i(t-s)B}(\sqrt{H}-B)e^{is\sqrt{H}}\|_{L^p([0,1],L^q)} \mathrm{d}s \\
&\lesssim \|u\|_{\mathcal{H}^{\sigma+\kappa}} + \int_0^1 \|(\sqrt{H}-B)e^{i(t-s)B}u\|_{\mathcal{H}^{\sigma+\kappa}} \mathrm{d}s \\
&\lesssim \|u\|_{\mathcal{H}^{\sigma+\kappa}}
\end{aligned}
$$

for any $\kappa > 0$. The proof is directly completed from

$$
\cos(t\sqrt{H}) = \frac{e^{it\sqrt{H}} + e^{-it\sqrt{H}}}{2}
$$

and

$$
\frac{\sin(\sqrt{H})}{\sqrt{H}} = \frac{e^{it\sqrt{H}} - e^{-it\sqrt{H}}}{2i\sqrt{H}}.
$$

□

Again, the inhomogeneous inequalities follow direclty and we ommit the proof.

Corollary 6.26. *Let p, q, σ be as in Theorem 6.25. Then we have the following bound*

$$
\Big\| \int_0^t \frac{\sin\big((t-s)\sqrt{H}\big)}{\sqrt{H}} f(s) \Big\|_{L^p([0,1],L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\sigma-1+\kappa}} \mathrm{d}s
$$

for $f \in L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})$.

Moreover, we have the analogous result for surfaces with boundary which is proved analogously by using Proposition 6.23 instead of Proposition 6.21.

Theorem 6.27. *Let M be a compact surface with boundary and $p, q \in [2, \infty)$ such that $p \le q$ and*

$$\sigma = \frac{5}{3} - \frac{2}{p} - \frac{4}{3q} \in (0, \alpha).$$

Then for any $\kappa > 0$, we have the bound

$$\Big\| \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} u_1 \Big\|_{L^p([0,1],L^q)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa}\times\mathcal{H}^{\sigma-1+\kappa}} + \|v\|_{L^1([0,1],\mathcal{H}^{\sigma-1+\kappa})}$$

and

$$\Big\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} v \Big\|_{L^p([0,1],L^q)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa}\times\mathcal{H}^{\sigma-1+\kappa}} + \|v\|_{L^1([0,1],\mathcal{H}^{\sigma-1+\kappa})}$$

for initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ and inhomogeneity $v \in L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})$.

6.2.2 – The special case of the two-dimensional torus

Finally, we discuss here how the procedure from the previous Section can be applied to get improved results on the torus. The only difference is that instead of Theorem 6.13, we have

$$\|e^{-itH}v\|_{L^4([0,1],L^4(\mathbb{T}^2))} \lesssim \|v\|_{\mathcal{H}^\kappa},$$

which was the central result of [59]. This implies improved L^q bounds on the eigenfunctions, we ommit the details since the proof is identical to before.

Proposition 6.28. *Consider the Anderson Hamiltonian H on the two-dimensional torus and let $q \in [4, \infty)$. Then we have the bound*

$$\|e_n\|_{L^q(\mathbb{T}^2)} \lesssim \sqrt{\lambda_n(\Xi)}^{1-\frac{4}{q}+\kappa} \sim (1+\sqrt{n})^{1-\frac{4}{q}+\kappa}.$$

Moreover, we get the bound for the spectral projector

$$\|\Pi_\lambda u\|_{L^q(\mathbb{T}^2)} \lesssim \sqrt{\lambda+1}^{1-\frac{4}{q}+\kappa} \|u\|_{L^2}.$$

Since the Weyl law holds in the same way in the particular case of the torus, we get the following improved result by the identical proof as before using Proposition 6.28.

Theorem 6.29. *Let $2 \le p \le q < \infty$ such that $q \ge 4$ and let*

$$\sigma = \frac{3}{2} - \frac{1}{p} - \frac{4}{q}.$$

Then for any $\kappa > 0$, we have the bound

$$\Big\| \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} u_1 \Big\|_{L^p_{t;[0,1]}L^q_M} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa}\times\mathcal{H}^{\sigma-1+\kappa}}$$

for initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$.

Remark : *Due to the nature of the bound from [59], there is no gain by allowing q to be smaller than 4. Indeed, the loss of derivative in that case is arbitrarily small and one does not gain anything by interpolating with the trivial L^2 bound.*

6.2.3 – Local well-posedness in low-regularity Sobolev spaces

Now we use the results from the previous Section to prove local well-posedness of stochastic multiplicative wave equations of the form

$$\left| \begin{array}{rcl} \partial_t^2 u + Hu & = & -u|u|^2 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1) \end{array} \right.$$

in a low-regularity regime both on general two-dimensional surfaces with boundary and in the special case of the two-dimensional torus. While we have the classical Sobolev embedding

$$\mathcal{H}^\nu \hookrightarrow L^{\frac{2}{1-\nu}}$$

for $\nu \in [0,1)$, we also make use of the following dual Sobolev bound

$$\forall \sigma \in (0,1], \quad L^{\frac{2}{2-\sigma}} \hookrightarrow \mathcal{H}^{\sigma-1}$$

which is true on general manifolds, see for example the book by Aubin [4]. Using this, we make a preliminary computation meant to show how far we get by using *only the Sobolev embedding result*. Then we will see how the bounds in Theorems 6.25 and 6.29 give better results on general manifolds and on the torus respectively. We first rewrite the equation under the mild formulation

$$u(t) = \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin\big((t-s)\sqrt{H}\big)}{\sqrt{H}}u(s)^3 \mathrm{d}s.$$

Then apply the dual Sobolev bound for $\sigma \in (0,1]$ and $p = \frac{2}{2-\sigma} \in (1,2]$ to get

$$\begin{aligned} \|u(t)\|_{\mathcal{H}^\sigma} \lesssim & \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,t],\mathcal{H}^{\sigma-1})} \\ \lesssim & \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,t],L^p)} \\ \lesssim & \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t],L^{\frac{2}{1-\sigma}})}\|u\|^2_{L^2([0,t],L^4)}, \end{aligned}$$

having applied Hölder with $\frac{1}{2} + \frac{1-\sigma}{2} = \frac{2-\sigma}{2}$. Finally, the Sobolev embedding gives

$$\|u(t)\|_{\mathcal{H}^\sigma} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|u\|^2_{L^2([0,t],\mathcal{H}^{\frac{1}{2}})}.$$

This can then lead to a solution by fixed point by choosing $\sigma \geq \frac{1}{2}$. Clearly this is can be improved by using more subtle bounds than the Sobolev embedding. The Strichartz inequalities from the previous section allow us to get local well-posedness below, this is the content of the following Theorems; As before we separately state the cases of surfaces without boundary, with boundary, and the special case of the torus which are proved in precisely the same way, just using Theorems 6.25, 6.27 and 6.29 respectively.

Theorem 6.30. *Let M be a compact surface without boundary and $\sigma \in (\frac{1}{4}, \frac{1}{2})$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ depending on the data such that there exists a unique solution*

$$u \in C\big([0,T], \mathcal{H}^\sigma\big) \cap L^{\frac{2}{1-\delta}}\big([0,T], L^4\big)$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0, u_1).

Proof : As usual, this is proved in a standard way using the Banach fixed point Theorem. Define the map

$$\Psi(u)(t) := \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin\big((t-s)\sqrt{H}\big)}{\sqrt{H}}u(s)^3 \mathrm{d}s.$$

For $t > 0$, we have as above

$$\begin{aligned}\|u(t)\|_{\mathcal{H}^\sigma} &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|u\|^2_{L^2([0,t],L^4)} \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + t^\delta\|u\|_{L^\infty([0,t],\mathcal{H}^\sigma)}\|u\|^2_{L^{\frac{2}{1-\delta}}([0,t],L^4)}\end{aligned}$$

using Hölder inequality in the last line for $\delta \in (0,1)$. We can apply Theorem 6.25 using with $p = \frac{2}{1-\delta}$

$$\begin{aligned}\big\|\Psi(u)\big\|_{L^{\frac{2}{1-\delta}}\left([0,T],L^4\right)} &\lesssim \|u_0\|_{\mathcal{H}^{1-\frac{1-\delta}{2}-\frac{1}{4}+\kappa}} + \|u_1\|_{\mathcal{H}^{-\frac{1-\delta}{2}-\frac{1}{4}+\kappa}} + \|u^3\|_{L^1([0,T],\mathcal{H}^{-\frac{1-\delta}{2}-\frac{1}{4}+\kappa})} \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,T],\mathcal{H}^{\sigma-1})}\end{aligned}$$

using that $\sigma > \frac{1}{4}$ and $\delta < \frac{\sigma}{2} - \frac{1}{2}$ gives $1 - \frac{1-\delta}{2} - \frac{1}{4} + \kappa \leq \sigma$ for $\kappa >$ small enough. Finally, we get

$$\big\|\Psi(u)\big\|_{L^{\frac{2}{1-\delta}}\left([0,T],L^4\right)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + T^\delta\|u\|_{L^\infty([0,T],\mathcal{H}^\sigma)}\|u\|^2_{L^{\frac{2}{1-\delta}}([0,T],L^4)}$$

as above. Thus we can get a fixed point in

$$C\big([0,T],\mathcal{H}^\sigma\big) \cap L^{\frac{2}{1-\delta}}\big([0,T],L^4\big)$$

in the usual way for $T > 0$ small enough.

□

In a completely analogous way we get the following result for the case of surfaces with boundary using the Strichartz estimates from Theorem 6.27.

Theorem 6.31. *Let M be a compact surface without boundary and $\sigma \in \left(\frac{1}{3},\frac{1}{2}\right)$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0,u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ depending on the data such that there exists a unique solution*

$$u \in C\big([0,T],\mathcal{H}^\sigma\big) \cap L^{\frac{2}{1-\delta}}\big([0,T],L^4\big)$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0,u_1).

In the special case of the two-dimensional torus, the range of regularities in which we obtain well-posedness is greater using the improved bound from Theorem 6.29 as one might expect. This is the following Theorem, we again omit the proof since it is similar to the previous one. Note that this result gives the maximal range of regularities for which the regularity measured in terms of H agrees with the usual regularity, see Proposition 6.10.

Theorem 6.32. *Let $M = \mathbb{T}^2$, $\sigma \in (0,\frac{1}{2})$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0,u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ such that there exists a unique solution*

$$u \in C\big([0,T],\mathcal{H}^\sigma(\mathbb{T}^2)\big) \cap L^{\frac{2}{1-\delta}}\big([0,T],L^4(\mathbb{T}^2)\big)$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0,u_1).

Chapter 7

Diffusions in disordered media

In this Chapter, we use the heat semigroup paracontrolled calculus to hint the study of two different random models of diffusions in disordered media. The first one is the polymer measure associated to the white noise on a two-dimensional surface which is formally given by

$$\mathbf{V}(\mathrm{d}X) = \frac{1}{Z_T} e^{-\frac{1}{2}\int_0^T \xi(X_s)\mathrm{d}s} \mathbf{W}(\mathrm{d}X)$$

with $\mathbf{W}$ the usual Wiener measure on $C([0,T],M)$. The Gibbsian formalism does not make sense here due to the singularity of the potential. Our approach is based on the intrinsic Feynman-Kac semigroup associated to the Anderson Hamiltonian. The second model is the Brox diffusion formally given by

$$\mathrm{d}X_t = \xi(X_t)\mathrm{d}t + \mathrm{d}B_t$$

with ξ a space white noise on the circle which corresponds to the derivative of a Brownian motion over $\mathbb{T}$. It is the continuous analogue of Sinai's random walk with infinitesimal generator formally given by

$$-\frac{1}{2}\Delta + \xi.\nabla$$

which falls into the range of singular stochastic operators. With the same methods used for the Anderson Hamiltonian and the random magnetic Laplacian, we construct this operator as an unbouded operator in L^2.

7.1 – The polymer measure with white noise potential

A natural way to proceed is first to consider a regularisation $(\xi_\varepsilon)_{\varepsilon>0}$ of the noise hence one needs to be able to deal with smooth potential. Thus we first deal with such potential and introduce the notion of intrinsic Feynman-Kac semigroup associated to a Schrödinger operator. We refer to the very great book [46] which mainly treat with the euclidian space $\mathbb{R}^d$ and applications to Quantum Field Theory.

In probability, the Schrödinger operator associated to a potential $V : M \to \mathbb{R}$ is

$$-\frac{1}{2}\Delta + V$$

where Δ is the Laplace-Beltrami operator on M. The factor $\frac{1}{2}$ is here to see a Schrödinger operator as a perturbation of the infinitesimal generator of the Brownian motion. If $V \in \mathcal{D}'(M)$, then

$$u \mapsto -\frac{1}{2}\Delta u + Vu$$

makes sense for $u \in C^\infty(M)$ and the behavior of the operator strongly depends on the different properties of its potential V. Under suitable conditions, one can prove that it is a self-adjoint operator bounded from below with pure point spectrum, the simpler example being the case of a smooth potential. In the unbounded setting, one requires in addition properties of V at infinity thus we will only consider bounded manifolds due to the lack of integrability of the white noise at infinity. As explained, we will restrict ourselves to $V \in C^\infty$ in this Section since we want to consider a regularisation $(\xi_\varepsilon)_{\varepsilon>0}$ of the noise. In this case, the operator

$$H = -\frac{1}{2}\Delta + V$$

is well-defined from C^∞ to L^2. A natural candidate as domain is the closure of the space

$$\{u \in C^\infty, Hu \in L^2\}$$

with respect to the norm domain

$$\|u\|_H := \|u\|_{L^2} + \|Hu\|_{L^2}.$$

In the case $V = 0$, this corresponds to the Sobolev space $\mathcal{H}^2$. This is also true for smooth potential V and the operator $(H, \mathcal{H}^2)$ is an unbounded operator in L^2, symmetric since the potential is real valued. It is even self-adjoint as a bounded from below symmetric operator. Finally, the injection

$$\mathcal{H}^2 \hookrightarrow L^2$$

is compact hence H has pure point spectrum. We denote as

$$\lambda_1(V) \leq \lambda_2(V) \leq \ldots \leq \lambda_n(V) \leq \ldots$$

its eigenvalues with multiplicity. Using a basis $(e_n)_{n\geq 1}$ of L^2 eigenfunctions of H, one can define $f(H)$ as a continuous operator in L^2 for any $f \in L^\infty\big([\lambda_1(V), +\infty)\big)$. This allows in particular to consider the semigroup associated to H

$$e^{-tH}u := \sum_{n\geq 1} e^{-t\lambda_n(V)}\langle u, e_n\rangle e_n$$

defined for any $t > 0$. Conversely, the semigroup can be used to deduce properties on H. For example, the Perron-Frobenius Theorem implies $\lambda_1(V) < \lambda_2(V)$ and positivity of e_1 given that e^{-H} is a positive operator in the sense

$$f, g \geq 0 \implies \langle f, e^{-tH} g\rangle > 0.$$

for all non-zero functions $f, g \in L^2$. The Feyman-Kac formula stated below will imply positivity of e^{-H} hence there exists a unique function Ψ of unit L^2-norm such that

$$H\Psi = \lambda_1(V)\Psi.$$

The function Ψ is called the ground state of H and $\lambda_1(V)$ is the ground state energy. In particular, $\Psi > 0$ and $V \in C^\infty$ implies $\Psi \in C^\infty$. The particular case $V = 0$ corresponds to the heat semigroup which is a central object in the theory of probability since it characterizes the law of the Brownian motion on the space of path. Indeed, the kernel of the heat semigroup corresponds to its probability transition as a continuous Markov process. Given a potential V, the semigroup of the associated Schrödinger operator is also related to the Brownian motion through the Feynman-Kac formula. It holds with various hypothesis on V, we suppose again V to be smooth for simplicity.

Theorem 7.1. *Let $V \in C^\infty(M)$. Then for all $f \in L^2(M)$ and $t > 0$, we have*

$$\left(e^{-tH} f\right)(x) = \mathbb{E}_x\left[e^{-\int_0^t V(B_s)\mathrm{d}s} f(B_t)\right]$$

where $\mathbb{E}_x$ denotes the expectation with respect to the Wiener measure starting at $x \in M$.

Proof : This is Theorem 3.30 in [46]. First, one can show that both sides of the equality are C^0-semigroups. For the left hand side, this comes from the properties of H. For the right hand side, it is bounded on $L^2(M)$ since the potential V is bounded from below. Moreoever, the time reversibility of the Brownian motion implies that it is symmetric and the semigroup properties comes from the Markov property. Finally, the strong continuity is simply obtained by dominated convergence. To conclud, one uses the Itô formula to prove that these two C^0-semigroups have the same generator.

□

The Feynman-Kac formula still holds for a large class of potential V, see Chapter 3 in [46] for different examples. While the heat semigroup gives the expectation of the Brownian motion

$$\mathbb{E}_x\big[f(B_t)\big] = \left(e^{\frac{t}{2}\Delta} f\right)(x),$$

one can ask about the existence of a process $(X_t)_{t\geq 0}$ such that

$$\mathbb{E}_x[f(X_t)] = \left(e^{-tH} f\right)(x)$$

for $t > 0$. In particular, this would imply

$$\left(e^{tH} 1\right)(x) = 1$$

for any $t > 0$ and $x \in M$, that is the semigroup associated to H would be conservative. In general, this is not the case thus one needs to modify the semigroup for this question to have a solution. Introduce the ground transform

$$\mathscr{U} : \left| \begin{array}{ccc} L^2(M,\mu) & \mapsto & L^2(M,\Psi^2\mu) \\ u & \mapsto & \Psi u \end{array} \right.$$

which is a unitary map since Ψ is positive and normalised in L^2 and define the operator

$$F = \mathscr{U}^{-1}\big(H - \lambda_1(V)\big)\mathscr{U}$$

with domain

$$\mathcal{D}_F = \left\{f \in L^2(M,\mu)\ ;\ \mathscr{U} f \in \mathcal{D}\right\}.$$

Then the semigroup associated to F is conservative by construction, it is called the intrinsic Feynman-Kac semigroup; this corresponds to the h-transform in terms of stochastic process. In the case $V \in C^\infty(M)$, the ground state Ψ is also smooth and we have the explicit formula

$$F = -\frac{1}{2}\Delta - \nabla \log \Psi \cdot \nabla$$

which corresponds to the infinitesimal generator of the process $(X_t)_{t>0}$ given by the SDE

$$\mathrm{d}X_t = \nabla(\log \Psi)\mathrm{d}t + \mathrm{d}B_t.$$

Thus, this gives an explicit process that answer our question in the case $V \in C^\infty(M)$. However, even when the operator K makes sense, the ground state might not be differentiable and the SDE is only formal. This is the case when $V = \xi$ is the space white noise in two dimensions and we need another way of constructing such process. In this setting, it is clear that $(X_t)_{t\geq 0}$ is a continuous Markov process with probability transition

$$K(t, x; s, y) = e^{-(t-s)F}(x, y)$$

for $0 \leq s \leq t$, that is the Schwartz Kernel of the semigroup associated to F. In particular, one can directly construct the law of X on path space with

$$\mathbf{V}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \int_{A_1 \times \ldots \times A_n} \left(\prod_{i=0}^{n-1} K(t_{i+1}, x_{i+1}; x_i, t_i) \right) \mathrm{d}x$$

for any $t_0 = 0 \leq t_1 \leq \ldots \leq t_n \leq T$, measurable sets $A_1, \ldots, A_n \subset M$ and $x_0 \in M$ the initial value. In order to prove that the measure is supported on the space of continuous function, one needs an estimate on the semigroup which is granted from the almost sure regularity of the Brownian motion and the Feynman-Kac formula. In particular, the measure is supported on paths of Hölder regularity $\frac{1}{2} - \kappa$ for any $\kappa > 0$. The Feynman-Kac formula also allows to relate the polymer measure with potential V to the intrinsic Feynman-Kac semigroup of $\Delta + V$ through density with respect to the Wiener measure $\mathbf{W}$.

To construct the polymer measure with white noise on a two-dimensional compact manifold, we want to use the same method. Since the Anderson Hamiltonian H is not a priori conservative, consider Ψ the Anderson groud state defined by

$$H\Psi = \lambda_1(\Xi)\Psi.$$

Since e^{-H} is positive, Perron-Frobenius Theorem again gives that Ψ is unique and positive. Using the intrinsic Feynman-Kac semigroup of the Anderson Hamiltonian and the formula

$$\mathbf{V}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \int_{A_1 \times \ldots \times A_n} \left(\prod_{i=0}^{n-1} K(t_{i+1}, x_{i+1}; x_i, t_i) \right) \mathrm{d}x,$$

one gets a measure on the set of functions from $\mathbb{R}$ to M, it formally correponds to the SDE with drift $\nabla(\log \Psi)$. In order to guarantee that this measure is supported on continuous path, one needs additionnal information. This is current investigation related to the study of SDEs with distributionnal drift, see for example [31, 32] and reference therein. In particular, the work [21] by Cannizzaro and Chouk studied the polymer measure with white noise using the KPZ equation and SDEs with noise-dependent distributional drift.

7.2 – The Brox diffusion and its generator

Given a potential $V : \mathbb{R} \to \mathbb{R}$, one can consider the diffusion

$$\mathrm{d}X_t = V(X_t)\mathrm{d}t + \mathrm{d}B_t$$

with infinitesimal generator

$$-\frac{1}{2}\Delta + V \cdot \nabla$$

which is an unbounded operator in L^2 for suitable potential V. We here consider $V = \xi$ a space white noise on the circle $\mathbb{T}$. This corresponds to the Brox diffusion which is formally described by the SDE

$$\mathrm{d}X_t = \xi(X_t)\mathrm{d}t + \mathrm{d}B_t$$

where the drift term $\xi(X_t)$ does not make sense since ξ is only a distribution, precisely it belongs to $\mathcal{C}^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$. This diffusion corresponds to a Brownian diffusion in a Brownian environment and was first studied by Brox in [16] on the full space. It is the continuous analogue of Sinai's random walk which can be described as follows. The environment is a collection $(\omega_x)_{x\in\mathbb{Z}}$ of independant and identically distributed random variables with values in $[0,1]$. Given the environment, Sinai's random walk $(S_n)_{n\geq 0}$ is a Markov chain starting at $S_0 = 0$ such that

$$\mathbb{P}(S_{n+1} = y \,|S_n = x) = \begin{cases} \omega_x & \text{if } y = x+1, \\ 1-\omega_x & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

An important random variable is

$$\eta_x := \log\left(\frac{1-\omega_x}{\omega_x}\right)$$

which is well-defined under the elliptic condition $\kappa < \omega_x < 1-\kappa$ for a given $\kappa > 0$. For example, the walk explodes to infinity for $\mathbb{E}[\eta_0] \neq= 0$ while it is recurrent for $\mathbb{E}[\eta_0] = 0$. In the deterministic case $\omega_x = p$, one recovers the condtion $p \neq \frac{1}{2}$ or $p = \frac{1}{2}$. In particular, the diffusion behavior is slowed by the disordered environment in the centered case

$$\lim_{n\to\infty} \frac{\sigma^2}{\log(n)^2} S_n = b_\infty$$

with $\sigma = \mathrm{Var}(\eta_0) < \infty$ and b_∞ a non-degenerate non-Gaussian random variable. To construct the continuous diffusion, Brox consider the generator under the form

$$-\frac{1}{2e^{-2B}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{e^{2B}}\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

where B is a Brownian motion obtained from integrating the white noise environment ξ, that is $B' = \xi$. Using the self-similarity of the Brownian motion B, he used Itô-McKean's construction and set

$$X_t = S^{-1}\big(B_{T^{-1}(t)}\big)$$

where $S(x)$ and $T(t)$ are given for $x \in \mathbb{R}$ and $t > 0$ by

$$S(x) := \int_0^x e^{2B_y}\mathrm{d}y \quad \text{and} \quad T(t) := \int_0^t e^{-4B_{S^{-1}(2B_s)}}\mathrm{d}s.$$

In this Section, we propre a different approach and construct directly the infinitesimal generator

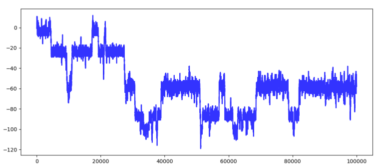
$$-\frac{1}{2}\Delta + \xi \cdot \nabla$$

as a singular stochastic operator. We consider the circle $\mathbb{T}$ because of the lack of integrability of the white noise to avoid the introduction of weight but losing the self-similarity property of the environment. For another approach to the Brox diffusion with the singular SPDEs approach, see the recent work [43] from Kremp and Perkowski where they solve the singular martingale problem on the full space with a Lévy noise.

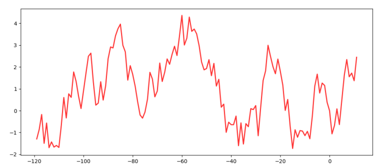
For Sinai's random walk, the slow-down of S_n is due to the presence of traps given by the wells of the potential

$$V_x = \begin{cases} \sum_{1\le y\le x} \eta_y & \text{if } x \ge 0 \\ -\sum_{-x+1\le y\le 0} \eta_y & \text{if } x < 0 \end{cases}$$

for $x \in \mathbb{Z}$. The idea is that the walk S_n escapes from a valley of V to get trapped in another one. This can be seen clearly with numerical simulation, for example in the simple case where $\omega_x \sim \mathcal{U}([\kappa, 1-\kappa])$ with $\kappa \in (0, \frac{1}{2})$.



Random walk $(S_n)_{n\ge 0}$



Random environment $(V_x)_{x\in\mathbb{Z}}$

Given the environment, $(S_n)_{n\ge 0}$ is a Markov chain with transition operator given by

$$Tf(x) = \omega_x f(x+1) + (1-\omega_x) f(x-1).$$

Thus the generator is

$$\begin{aligned} Lf(x) &= (T-I)f(x) \\ &= \omega_x f(x+1) + (1-\omega_x) f(x-1) - f(x) \\ &= \frac{1}{2}\big[f(x+1) + f(x-1) - 2f(x)\big] + \big(\omega_x - \frac{1}{2}\big)\big[f(x+1) - f(x-1)\big] \\ &= \frac{1}{2}\Delta f(x) + \xi_x \cdot \nabla f(x) \end{aligned}$$

with Δ a discrete Laplacian, ∇ a discrete derivative and

$$\xi := \left(\omega_x - \frac{1}{2}\right)_{x\in\mathbb{Z}}$$

the centered random environment. This hints

$$H := \frac{1}{2}\Delta + \xi \cdot \nabla$$

as the formal infinitesimal generator of the Brox diffusion. As explained, the noise ξ belongs to $\mathcal{C}^{\alpha-2}$ for any $\alpha < \frac{3}{2}$ hence the product $\xi \cdot \nabla u$ does not belong to L^2 for a generic smooth function u. It thus falls in the range of singular stochastic operators and we want to construct a domain $\mathcal{D}$ such that $(H,\mathcal{D})$ is a self-adjoint unbounded operator in L^2 with dense domain, maybe after a renormalisation procedure. For $u\in C^\infty$ such that $Hu\in L^2$, we have

$$-\frac{1}{2}\Delta u = Hu - \xi\cdot\nabla u \in \mathcal{H}^{\alpha-2}$$

hence u is expected to belong to $\mathcal{H}^\alpha$. Thus consider $u\in\mathcal{H}^\alpha$, we have

$$-\frac{1}{2}\Delta u = Hu - \xi\cdot\nabla u \in \mathcal{H}^{\alpha-2}$$

where the product $\nabla u\cdot\xi$ is singular since $2\alpha-3<0$ but formally belongs to $\mathcal{H}^{\alpha-2}$ since $\alpha-1>0$. As for the Anderson Hamiltonian, the idea is to cancel the singular part of the product with noise with roughness in the Laplacian term through a suitable paracontrolled expansion. Since the roughest part is given by the paraproduct of ξ by ∇u, we consider

$$u = \widetilde{\mathsf{P}}_{\nabla u}X + u^\sharp$$

with $u^\sharp\in\mathcal{H}^2$ and

$$X = 2\Delta^{-1}\xi\in\mathcal{C}^\alpha.$$

This formally gives

$$\begin{aligned}
Hu &= -\frac{1}{2}\Delta u + \xi\cdot\nabla u\\
&= -\frac{1}{2}\mathsf{P}_{\nabla u}\Delta X - \frac{1}{2}\Delta u^\sharp + \mathsf{P}_{\nabla u}\xi + \mathsf{P}_\xi\nabla u + \mathsf{\Pi}(\nabla u,\xi)\\
&= -\frac{1}{2}\Delta u^\sharp + \mathsf{P}_\xi\nabla u + \mathsf{\Pi}(\nabla u,\xi)
\end{aligned}$$

where the resonant term $\mathsf{\Pi}(\nabla u,\xi)$ is ill-defined but formally belongs to $\mathcal{H}^{2\alpha-3}$. To deal with the singularity, we write

$$\begin{aligned}
\mathsf{\Pi}(\nabla u,\xi) &= \mathsf{\Pi}(\nabla\widetilde{\mathsf{P}}_{\nabla u}X,\xi) + \mathsf{\Pi}(\nabla u^\sharp,\xi)\\
&= \nabla u\mathsf{\Pi}(\nabla X,\xi) + \mathsf{C}_\nabla(\nabla u,X,\xi) + \mathsf{\Pi}(\nabla u^\sharp,\xi)
\end{aligned}$$

where the correcteor is given by

$$\mathsf{C}_\nabla(a_1,a_2,b) := \mathsf{\Pi}(\nabla\widetilde{\mathsf{P}}_{a_1}a_2,b) - a_1\mathsf{\Pi}(\nabla a_2,b).$$

It satifies analogue continuity estimates as the classical corrector C or the corrector C_∂ used to solve the KPZ equation. The construction of the singular term $\mathsf{\Pi}(\nabla X,\xi)$ is done through a renormalisation procedure as an element of its natural space $\mathcal{C}^{2\alpha-3}$. With this, H is then a well-defined operator from

$$\{u\in L^2\ ;\ u-\widetilde{\mathsf{P}}_{\nabla u}X\in\mathcal{H}^2\}$$

with values in $\mathcal{H}^{2\alpha-3}$ which is however not contained in L^2 since $2\alpha-3$. This is not surprising since the product was singular and a second order expansion is needed. The rough term that we want to cancel is $\mathsf{P}_{\nabla u}\mathsf{\Pi}(\nabla X,\xi)$ thus we consider the second order expansion

$$u = \widetilde{\mathsf{P}}_{\nabla u}X_1 + \widetilde{\mathsf{P}}_{\nabla u}X_2 + u^\sharp$$

with

$$X_1 := 2\Delta^{-1}\xi \in \mathcal{C}^\alpha \quad \text{and} \quad X_2 := 2\Delta^{-1}\Pi(\nabla X, \xi) \in \mathcal{C}^{2\alpha-1}$$

given the enhanced noise

$$\Xi := \big(\xi, \Pi(\nabla X_1, \xi)\big) \in \mathcal{X}^\alpha := \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha-3}.$$

Definition 7.2. *We define the space $\mathcal{D}_\Xi$ of functions paracontrolled by Ξ as*

$$\mathcal{D}_\Xi := \{u \in L^2;\ u^\sharp := u - \widetilde{\mathsf{P}}_{\nabla u} X_1 - \widetilde{\mathsf{P}}_{\nabla u} X_2 \in \mathcal{H}^2\}.$$

This is done as for the Anderson Hamiltoniann in Chapter 4 and the random magnetic Laplacian in Chapter 5. The renormalisation probabilistic procedure to construct the enhanced noise Ξ is needed to deal with the singularity as in the other Chapters. Since the operator is not symmetric even for smooth potential V, its study requires new ideas from the classical study of unbounded operator of this type but one can expect the general idea to be the same. In particular, H should be the resolvent-limit of

$$H_\varepsilon := \frac{1}{2}\Delta + (\xi_\varepsilon - c_\varepsilon) \cdot \nabla$$

as ε goes to 0 for some appropriated renormalisation constant c_ε. As explained, the Brox diffusion is formally described by

$$\mathrm{d}X_t = \xi(X_t)\mathrm{d}t + \mathrm{d}B_t$$

where the drift $\xi(X_t)$ does not make sense. Heuristically, its infinitesimal generator should correspond to the singular stochastic operator

$$H = -\frac{1}{2}\Delta + \xi \cdot \nabla$$

constructed in the previous Section. While Brox construction of the process relied on the Ito-Mckean construction of a Feller diffusions, this gives a new approach where one constructs directly the law of the process on path space using the semigroup associated to $(H, \mathcal{D}_\Xi)$ as done for the polymer measure with white noise potential. Indeed, one has

$$\mathbb{E}_x[f(X_t)] = \big(e^{-tH} f\big)(x)$$

for any $x \in \mathbb{T}$ and $f \in L^\infty$. Since the process $(X_t)_{t\geq 0}$ is a Markov process, it is enough to specify its probability transitions given by

$$\mathbb{P}(X_t \in \mathrm{d}x\ |X_s \in \mathrm{d}y) = e^{-(t-s)H}(x,y)$$

for any $0 \leq s \leq t$ and $x, y \in \mathbb{T}$. This gives a measure on the set of function from $\mathbb{R}$ to $\mathcal{T}$ and one needs additionnal information to guarantee that this measure is supported on continuous path. As for the polymer measure with white noise potential, this is related to SDEs with time-independent distributionnal drift, see again [31, 32] and references therein. In the case of the Brox diffusion, Kremp and Perkowski recently solved the singular martingale problem associated in the case of a Lévy process driving the SDE, see [43]. The properties of the process that can be deduced from this construction in the spirit of this paper are work under investigation.

Chapter A

Paracontrolled calculus toolbox

A.1 – Approximation operators

We describe in this Appendix technical estimates needed in our continuous setting analogue of the discrete Paley-Littlewood decomposition. The following Proposition is the analogue of the inclusions of ℓ^p spaces.

Proposition A.1. *Let $p, q_1, q_2 \in [1,\infty]$ with $q_1 \leq q_2$. For $f \in L^p$ and $\alpha \in \mathbb{R}$, we have*

$$\left\| t^{-\frac{\alpha}{2}} \|Q_t f\|_{L^p_x} \right\|_{L^{q_2}(t^{-1}\mathrm{d}t)} \lesssim \left\| t^{-\frac{\alpha}{2}} \|Q_t f\|_{L^p_x} \right\|_{L^{q_1}(t^{-1}\mathrm{d}t)}.$$

Proof : We prove $\|\cdot\|_{L^\infty(t^{-1}\mathrm{d}t)} \lesssim \|\cdot\|_{L^q(t^{-1}\mathrm{d}t)}$ for any $q \in [1,\infty)$ and the result follows from duality. To get this, we use

$$Q_t = 2\int_{\frac{t}{2}}^t Q_s \left(\frac{t}{s}\right)^{a+1} P^{(c)}_{t-s}\frac{\mathrm{d}s}{s}$$

for any $Q \in \mathsf{StGC}^a$ and $t \in (0,1]$ which yields

$$\|Q_t f\|_{L^p} \lesssim \int_{\frac{t}{2}}^t \|Q_s f\|_{L^p}\frac{\mathrm{d}t}{t} \lesssim \left(\int_{\frac{t}{2}}^t \|Q_s f\|^q_{L^p}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

□

One needs the following bound to keep an accurate track of the constant in different estimates.

Lemma A.2. *Let $r > 0$ and $\alpha \in (-r,r)$. We have*

$$\int_0^\infty \left(\frac{u}{1+u^2}\right)^r u^\alpha \frac{\mathrm{d}u}{u} \leq \frac{2r}{r^2-\alpha^2}.$$

Proof : Since

$$1 = \frac{1+u^2}{1+u^2} = \frac{1}{1+u^2} + \frac{u^2}{1+u^2}$$

and $u \geq 0$, both terms are bounded by 1 and we have

$$\int_0^\infty \left(\frac{u}{1+u^2}\right)^r u^\alpha \frac{\mathrm{d}u}{u} = \int_0^1 \left(\frac{u}{1+u^2}\right)^r u^\alpha \frac{\mathrm{d}u}{u} + \int_1^\infty \left(\frac{u}{1+u^2}\right)^r u^\alpha \frac{\mathrm{d}u}{u}$$
$$\leq \frac{1}{r+\alpha} + \frac{1}{r-\alpha}$$

hence the bound.

□

The next Lemma describes the localisation of the cancellation in our continuous context, including the dependance on $s > 0$.

Lemma A.3. *Let $r > 0$ and $\alpha \in (-r, r)$. Given any $q \in [1, \infty]$, we have*

$$\left\| u^{-\alpha} \int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r f(t) \frac{\mathrm{d}t}{t} \right\|_{L^q(u^{-1}\mathrm{d}u)} \leq \frac{2r}{r^2-\alpha^2} \left\| u^{-\alpha} f(u) \right\|_{L^q(u^{-1}\mathrm{d}u)}.$$

We also have

$$\left\| u^{-\alpha} \int_0^s \left(\frac{tu}{(t+u)^2}\right)^r f(t) \frac{\mathrm{d}t}{t} \right\|_{L^q(u^{-1}\mathrm{d}u)} \leq \frac{2r}{r^2-\alpha^2} s^{\beta-\alpha} \left\| u^{-\beta} f(u) \right\|_{L^q(u^{-1}\mathrm{d}u)}$$

for any $s > 0$ and $\beta \in (\alpha, r)$.

Proof: For $q = \infty$, we have

$$\left| \int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r f(t) \frac{\mathrm{d}t}{t} \right| \leq \|t^{-\alpha} f(t)\|_{L^\infty} \int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r t^\alpha \frac{\mathrm{d}t}{t}$$
$$\leq \left(\int_0^\infty \left(\frac{v}{1+v^2}\right)^r v^\alpha \frac{\mathrm{d}v}{v} \right) u^\alpha \|t^{-\alpha} f(t)\|_{L^\infty}$$
$$\leq \frac{2r}{r^2-\alpha^2} u^\alpha \|t^{-\alpha} f(t)\|_{L^\infty}$$

which yields the result. For $q = 1$, we have

$$\int_0^1 u^{-\alpha} \left| \int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r f(t) \frac{\mathrm{d}t}{t} \right| \frac{\mathrm{d}u}{u} \leq \int_0^1 \left(\int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r u^{-\alpha} \frac{\mathrm{d}u}{u} \right) |f(t)| \frac{\mathrm{d}t}{t}$$
$$\leq \left(\int_0^\infty \left(\frac{v}{1+v^2}\right)^r v^\alpha \frac{\mathrm{d}v}{v} \right) \int_0^1 t^{-\alpha} |f(t)| \frac{\mathrm{d}t}{t}$$
$$\leq \frac{2r}{r^2-\alpha^2} \int_0^1 t^{-\alpha} |f(t)| \frac{\mathrm{d}t}{t}.$$

The result then follows for any $q \in (1, \infty)$ by interpolation. For the dependance with respect to s, we also interpolate between $q = 1$ and $q = \infty$ and conclud with

$$\left| \int_0^s \left(\frac{tu}{(t+u)^2}\right)^r f(t) \frac{\mathrm{d}t}{t} \right| \leq \|t^{-\beta} f(t)\|_{L^\infty} \int_0^s \left(\frac{tu}{(t+u)^2}\right)^r t^\beta \frac{\mathrm{d}t}{t}$$
$$\leq s^{\beta-\alpha} \|t^{-\beta} f(t)\|_{L^\infty} \int_0^s \left(\frac{tu}{(t+u)^2}\right)^r t^\alpha \frac{\mathrm{d}t}{t}$$
$$\leq \frac{2r}{r^2-\alpha^2} s^{\beta-\alpha} u^\alpha \|t^{-\alpha} f(t)\|_{L^\infty}$$

and

$$\int_0^1 u^{-\alpha}\left|\int_0^s \left(\frac{tu}{(t+u)^2}\right)^r f(t)\frac{\mathrm{d}t}{t}\right|\frac{\mathrm{d}u}{u} \leq \int_0^s \left(\int_0^1 \left(\frac{tu}{(t+u)^2}\right)^r u^{-\alpha}\frac{\mathrm{d}u}{u}\right)|f(t)|\frac{\mathrm{d}t}{t}$$
$$\leq \frac{2r}{r^2-\alpha^2}\int_0^s t^{-\alpha}|f(t)|\frac{\mathrm{d}t}{t}$$
$$\leq \frac{2r}{r^2-\alpha^2}s^{\beta-\alpha}\int_0^1 t^{-\beta}|f(t)|\frac{\mathrm{d}t}{t}.$$

□

Finally, we have the following estimate for integrals.

Lemma A.4. *Given any $\alpha > 0$ and $q \in [1,\infty]$, we have*

$$\left\|u^{-\frac{\alpha}{2}}\int_0^u f(t)\frac{\mathrm{d}t}{t}\right\|_{L^q(u^{-1}\mathrm{d}u)} \leq \frac{2}{\alpha}\|u^{-\frac{\alpha}{2}}f(u)\|_{L^q(u^{-1}\mathrm{d}u)}.$$

Proof : We proceed again by interpolation proving the estimate for $q=\infty$ and $q=1$. Using that $\alpha > 0$, we have

$$\left|\int_0^u f(t)\frac{\mathrm{d}t}{t}\right| \leq \|t^{-\frac{\alpha}{2}}f(t)\|_{L^\infty}\int_0^u t^{\frac{\alpha}{2}}\frac{\mathrm{d}t}{t} \leq \frac{2}{\alpha}u^{\frac{\alpha}{2}}\|t^{-\frac{\alpha}{2}}f(t)\|_{L^\infty}$$

and

$$\int_0^1 u^{-\frac{\alpha}{2}}\left|\int_0^u f(t)\frac{\mathrm{d}t}{t}\right|\frac{\mathrm{d}u}{u} \leq \int_0^1\left(\int_t^1 u^{-\frac{\alpha}{2}}\frac{\mathrm{d}u}{u}\right)|f(t)|\frac{\mathrm{d}t}{t} \leq \frac{2}{\alpha}\int_0^1 t^{-\frac{\alpha}{2}}|f(t)|\frac{\mathrm{d}t}{t}.$$

□

A.2 – Estimates on paraproducts, correctors and commutators

We give in this Appendix proofs of estimates needed in paracontrolled calculus. We shall first prove the estimates for the paraproduct P and resonant operator $\mathsf{\Pi}$ in Sobolev spaces. It works as for Hölder spaces with L^2 estimates instead of L^∞.

Proposition A.5. *Let $\alpha,\beta \in (-2b,2b)$ be regularity exponent.*

- *If $\alpha > 0$, then $(f,g)\mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha\times\mathcal{H}^\beta$ to $\mathcal{H}^\beta$ and from $\mathcal{H}^\alpha\times\mathcal{C}^\beta$ to $\mathcal{H}^\beta$.*
- *If $\alpha < 0$, then $(f,g)\mapsto \mathsf{P}_f g$ is continuous from $\mathcal{C}^\alpha\times\mathcal{H}^\beta$ to $\mathcal{H}^{\alpha+\beta}$ and from $\mathcal{H}^\alpha\times\mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*
- *If $\alpha+\beta > 0$, then $(f,g)\mapsto \mathsf{\Pi}(f,g)$ is continuous from $\mathcal{H}^\alpha\times\mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*

Proof : Let $f \in \mathcal{H}^\alpha$ and $g \in \mathcal{C}^\beta$ with $\alpha < 0$. We want to compute the regularity $\mathcal{H}^{\alpha+\beta}$ of $\mathsf{P}_f g$ hence let $Q \in \mathsf{StGC}^r$ with $r > |\alpha+\beta|$. Recall that $\mathsf{P}_f g$ is a linear combination of terms of the form

$$\int_0^1 Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$. Given $s \in (0,1]$, we have

$$\begin{aligned}\left\|\int_0^1 Q_s Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x} &\lesssim \int_0^1 \left(\frac{ts}{(t+s)^2}\right)^{\frac{r}{2}} \left\|P_t f \cdot Q_t^2 g\right\|_{L^2_x}\frac{\mathrm{d}t}{t}\\ &\lesssim \|g\|_{\mathcal{C}^\beta}\int_0^1 \left(\frac{ts}{(t+s)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\|P_t f\|_{L^2_x}\frac{\mathrm{d}t}{t}.\end{aligned}$$

This yields

$$\begin{aligned}&\left\|s^{-\frac{\alpha+\beta}{2}}\left\|\int_0^1 Q_s Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x}\right\|_{L^2(s^{-1}\mathrm{d}s)}\\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\left\|s^{-\frac{\alpha+\beta}{2}}\int_0^1 \left(\frac{ts}{(t+s)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\|P_t f\|_{L^2_x}\frac{\mathrm{d}t}{t}\right\|_{L^2(s^{-1}\mathrm{d}s)}\\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\left\|s^{-\frac{\alpha}{2}}\|P_s f\|_{L^2_x}\right\|_{L^2(s^{-1}\mathrm{d}s)}\\ &\qquad\lesssim \|f\|_{\mathcal{H}^\alpha}\|g\|_{\mathcal{C}^\beta}\end{aligned}$$

where we used that $\alpha < 0$ since P can encode no cancellation and this complete the proof for the third estimate. The proofs for the other estimates on $\mathsf{P}_f g$ are similar and we only give the details for the resonant term. Let $Q \in \mathsf{StGC}^r$ with $r > |\alpha+\beta|$ and recall that $\mathsf{\Pi}(f,g)$ is a linear combination of terms

$$\int_0^1 P_t^\bullet\left(Q_t^1 f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $Q^1, Q^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$. Given $s \in (0,1]$, we have

$$\left\|\int_0^1 Q_s P_t^\bullet\left(Q_t^1 f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x} \lesssim \int_0^s \|Q_t^1 f \cdot Q_t^2 g\|_{L^2_x}\frac{\mathrm{d}t}{t} + \int_s^1 \left(\frac{s}{t}\right)^{\frac{r}{2}}\left\|Q_t^1 f \cdot Q_t^2 g\right\|_{L^2_x}\frac{\mathrm{d}t}{t}$$

and the result follows again from the Lemmas using that $\alpha+\beta > 0$.

□

The dependance of $\widetilde{\mathsf{P}}^s$ with respect to s in given in the following Proposition.

Proposition A.6. *Let $s \in (0,1)$ and a regularity exponent $\beta \in (0,1)$. Given $g \in \mathcal{C}^\beta$, we have*

$$\|f \mapsto \widetilde{\mathsf{P}}_f^s g\|_{L^2 \to \mathcal{H}^\gamma} \lesssim \frac{s^{\frac{\beta-\gamma}{4}}}{\beta-\gamma}\|g\|_{\mathcal{C}^\beta}$$

for any $\gamma \in [0,\beta)$.

Proof : Given $f \in L^2$ and $\gamma \in [0,\beta)$, we want to bound the $\mathcal{H}^\gamma$ norm of $\widetilde{\mathsf{P}}^s_f g$ hence let $Q \in \mathsf{StGC}^r$ with $r > |\gamma|$. Recall that $\widetilde{\mathsf{P}}^s_f g$ is a linear combination of terms of the form

$$\int_0^s \widetilde{Q}_t^{1\bullet}\left(P_t f \cdot \widetilde{Q}_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $\widetilde{Q}^1 \in \mathsf{GC}^{\frac{b}{2}-2}, \widetilde{Q}^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$. Given $u \in (0,1]$, we have

$$\begin{aligned}\left\|\int_0^s Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x} &\lesssim \int_0^s \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} \|P_t f \cdot Q_t^2 g\|_{L^2_x}\frac{\mathrm{d}t}{t} \\ &\lesssim \|g\|_{\mathcal{C}^\beta}\int_0^s \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\|P_t f\|_{L^2_x}\frac{\mathrm{d}t}{t}.\end{aligned}$$

This yields

$$\begin{aligned}&\left\|u^{-\frac{s}{2}}\left\|\int_0^s Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x}\right\|_{L^2(u^{-1}\mathrm{d}u)} \\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\left\|u^{-\frac{\gamma}{2}}\int_0^s \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\|P_t f\|_{L^2_x}\frac{\mathrm{d}t}{t}\right\|_{L^2(u^{-1}\mathrm{d}u)} \\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\frac{4r}{r^2-\gamma^2}s^{\frac{\beta'-\gamma}{2}}\|u^{-\frac{\beta'-\beta}{2}}\|P_u f\|_{L^2}\|_{L^2(u^{-1}\mathrm{d}u)} \\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\frac{4r}{r^2-\gamma^2}s^{\frac{\beta'-\gamma}{2}}\frac{2}{k+\beta-\beta'}\|f\|_{\mathcal{H}^{\beta'-\beta}} \\ &\qquad\lesssim \frac{\|g\|_{\mathcal{C}^\beta}}{1-\beta}\frac{s^{\frac{\beta'-\gamma}{2}}}{k+\beta-\beta'}\|f\|_{\mathcal{H}^{\beta'-\beta}}\end{aligned}$$

for any $\beta' \in (\gamma,\beta)$ and $P \in \mathsf{StGC}^k$ using that $r \geq 1$. For $k \geq 1$, one can take $\beta' = \beta$ and get

$$\left\|u^{-\frac{\gamma}{2}}\left\|\int_0^s Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x}\right\|_{L^2(u^{-1}\mathrm{d}u)} \lesssim \frac{s^{\frac{\beta-\gamma}{2}}}{1-\beta}\|g\|_{\mathcal{C}^\beta}\|f\|_{L^2}.$$

For $k = 0$, we have

$$\left\|u^{-\frac{\gamma}{2}}\left\|\int_0^s Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x}\right\|_{L^2(u^{-1}\mathrm{d}u)} \lesssim \frac{\|g\|_{\mathcal{C}^\beta}}{1-\beta}\frac{s^{\frac{\beta'-\gamma}{2}}}{\beta-\beta'}\|f\|_{L^2}$$

hence taking $\beta' = \frac{\gamma+\beta}{2}$ yields

$$\left\|u^{-\frac{\gamma}{2}}\left\|\int_0^s Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L^2_x}\right\|_{L^2(u^{-1}\mathrm{d}u)} \lesssim \frac{s^{\frac{\beta-\gamma}{4}}}{(1-\beta)(\beta-\gamma)}\|g\|_{\mathcal{C}^\beta}\|f\|_{L^2}.$$

□

Proposition A.7. *Let $s \in (0,1)$ and a regularity exponent $\beta < 2$. Given $g \in \mathcal{C}^\beta$, we have*

$$\|(\widetilde{\mathsf{P}}_f - \widetilde{\mathsf{P}}^s_f)g\|_{\mathcal{H}^2} \lesssim s^{\frac{\beta-2}{2}}\|f\|_{L^2}\|g\|_{\mathcal{C}^\beta}$$

for any $f \in L^2$.

Proof : Given $f \in L^2$, we want to bound the $\mathcal{H}^2$ norm of $(\widetilde{\mathsf{P}}_f - \widetilde{\mathsf{P}}_f^s)g$ hence let $Q \in \mathsf{StGC}^r$ with $r > 2$. It is a linear combination of terms

$$\int_s^1 \widetilde{Q}_t^{1\bullet}\left(P_t f \cdot \widetilde{Q}_t^2 g\right)\frac{\mathrm{d}t}{t}$$

with $\widetilde{Q}^1 \in \mathsf{GC}^{\frac{b}{2}-2}, \widetilde{Q}^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $P \in \mathsf{StGC}^{[0,b]}$. Given $u \in (0,1]$, we have

$$\begin{aligned}\left\|\int_s^1 Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\right\|_{L_x^2} &\lesssim \int_s^1 \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} \left\|P_t f \cdot Q_t^2 g\right\|_{L_x^2}\frac{\mathrm{d}t}{t}\\ &\lesssim \|f\|_{L^2}\|g\|_{\mathcal{C}^\beta}\int_s^1 \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\frac{\mathrm{d}t}{t}\end{aligned}$$

using that $\|P_t f\|_{L^2} \lesssim \|f\|_{L^2}$. This yields

$$\begin{aligned}&\left\|u^{-1}\right\|\int_s^1 Q_u Q_t^{1\bullet}\left(P_t f \cdot Q_t^2 g\right)\frac{\mathrm{d}t}{t}\Big\|_{L_x^2}\Big\|_{L^2(u^{-1}\mathrm{d}u)}\\ &\qquad\lesssim \|g\|_{\mathcal{C}^\beta}\left\|u^{-1}\int_s^1 \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} t^{\frac{\beta}{2}}\frac{\mathrm{d}t}{t}\right\|_{L^2(u^{-1}\mathrm{d}u)}\\ &\qquad\lesssim s^{\frac{\beta-2}{2}}\|f\|_{L^2}\|g\|_{\mathcal{C}^\beta}\end{aligned}$$

and the proof is complete.

$\square$

Proposition A.8. *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta < 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \beta > 0,$$

then $(a_1, a_2, b) \mapsto \mathsf{C}(a_1, a_2, b)$ extends in a unique bilinear operator from $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^\beta$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$ and from $\mathcal{H}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^\beta$ to $\mathcal{H}^{\alpha_1+\alpha_2+\beta}$.

Proof : We first consider $(a_1, a_2, b) \in \mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^\beta$. We want to compute the regularity of

$$\mathsf{C}(a_1, a_2, b) = \mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{a_1}a_2, b\big) - a_1\mathsf{\Pi}(a_2, b)$$

using a family Q of StGC^r with $r > |\alpha_1+\alpha_2+\beta|$. Recall that a term $\mathsf{\Pi}(a,b)$ can be written as a linear combination of terms of the form

$$\int_0^1 P_t^{1\bullet}(Q_t^1 a \cdot Q_t^2 b)\frac{dt}{t},$$

while $\widetilde{\mathsf{P}}_b a$ is a linear combination of terms of the form

$$\int_0^1 \widetilde{Q}_t^{3\bullet}\big(P_t^2 b \cdot \widetilde{Q}_t^4 a\big)\frac{dt}{t}$$

with $Q^1, Q^2, \widetilde{Q}^4 \in \mathsf{StGC}^{\frac{b}{2}}$, $\widetilde{Q}^3 \in \mathsf{GC}^{\frac{b}{2}-2}$ and $P^1, P^2 \in \mathsf{StGC}^{[0,b]}$. For $P^2 \in \mathsf{StGC}^{[1,b]}$, we already have the correct regularity since

$$\begin{aligned}&\int_0^1\int_0^1 Q_u P_t^{1\bullet}\left(Q_t^1\widetilde{Q}_s^{3\bullet}\left(P_s^2 a_1 \cdot \widetilde{Q}_s^4 a_2\right)\cdot Q_t^2 b\right)\frac{ds}{s}\frac{dt}{t}\\ &\qquad\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \int_0^1\int_0^1 \left(\frac{ut}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{ts}{(s+t)^2}\right)^{\frac{b}{2}} s^{\frac{\alpha_1+\alpha_2}{2}} t^{\frac{\beta}{2}}\frac{ds}{s}\frac{dt}{t}\\ &\qquad\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\, u^{\frac{\alpha_1+\alpha_2+\beta}{2}}\end{aligned}$$

using that $\alpha_1 \in (0,1)$. We consider $P^2 \in \mathsf{StGC}^0$ for the remainder of the proof. For all $x \in M$, we have

$$\begin{aligned}
\mathsf{C}\big(a_1, a_2, b\big)(x) &= \mathsf{\Pi}\left(\widetilde{\mathsf{P}}_{a_1} a_2, b\right)(x) - a_1(x) \cdot \mathsf{\Pi}(a_2, b)(x) \\
&= \mathsf{\Pi}\left(\widetilde{\mathsf{P}}_{a_1} a_2 - a_1(x) \cdot a_2, b\right)(x) \\
&\simeq \mathsf{\Pi}\left(\widetilde{\mathsf{P}}_{a_1 - a_1(x)} a_2, b\right)(x),
\end{aligned}$$

since $\mathsf{\Pi}$ is bilinear and $a_1(x)$ is a scalar and $\widetilde{\mathsf{P}}_1 a_1 = a_1$ up to smooth terms. Thus we only have to consider a linear combination of terms of the form

$$\int_0^1 \int_0^1 P_t^{1\bullet}\Big(Q_t^1 \widetilde{Q}_s^{3\bullet}\left(\big(P_s^2 a_1 - a_1(x)\big) \cdot \widetilde{Q}_s^4 a_2\right) \cdot Q_t^2 b\Big)(x)\, \frac{ds}{s}\frac{dt}{t}$$

using that $\displaystyle\int_0^1 \widetilde{Q}_s^{3\bullet} \widetilde{Q}_s^4 \frac{ds}{s} = \mathrm{Id}$ up to smooth terms. This gives $\big(Q_u \mathsf{C}(a_1, a_2, b)\big)(x)$ as a linear combination of terms of the form

$$\begin{aligned}
&\int K_{Q_u}(x,x') P_t^{1\bullet}\Big(Q_t^1 \widetilde{Q}_s^{3\bullet}\left(\big(P_s^2 a_1 - a_1(x')\big) \cdot \widetilde{Q}_s^4 a_2\right) \cdot Q_t^2 b\Big)(x')\, \frac{ds}{s}\frac{dt}{t} \nu(dx') \\
&= \int K_{Q_u}(x,x') K_{P_t^{1\bullet}}(x',x'')\Big(Q_t^1 \widetilde{Q}_s^{3\bullet}\left(\big(P_s^2 a_1 - a_1(x'')\big) \cdot \widetilde{Q}_s^4 a_2\right) \cdot Q_t^2 b\Big)(x'')\frac{ds}{s}\frac{dt}{t}\nu(dx')\nu(dx'') \\
&\quad + \int\int_0^u K_{Q_u}(x,x') K_{P_t^{1\bullet}}(x',x'')\Big(a_1(x'') - a_1(x')\Big)\left(Q_t^1 a_2 \cdot Q_t^2 b\right)(x'')\frac{dt}{t}\nu(dx')\nu(dx'') \\
&\quad + \int\int_u^1 K_{Q_u}(x,x') K_{P_t^{1\bullet}}(x',x'')\Big(a_1(x'') - a_1(x')\Big)\left(Q_t^1 a_2 \cdot Q_t^2 b\right)(x'')\frac{dt}{t}\nu(dx')\nu(dx'') \\
&=: A + B + C.
\end{aligned}$$

The term A is bounded using cancellations properties. We have

$$\begin{aligned}
|A| &= \int K_{Q_u P_t^{1\bullet}}(x,x')\Big(Q_t^1 \widetilde{Q}_s^{3\bullet}\left(\big(P_s^2 a_1 - a_1(x')\big) \cdot \widetilde{Q}_s^4 a_2\right) \cdot Q_t^2 b\Big)(x')\frac{ds}{s}\frac{dt}{t}\nu(dx') \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \Bigg(\int_0^u\int_0^1 \left(\frac{st}{(s+t)^2}\right)^{\frac{b}{2}} (s+t)^{\frac{\alpha_1}{2}} s^{\frac{\alpha_2}{2}} t^{\frac{\beta}{2}} \frac{ds}{s}\frac{dt}{t} \\
&\qquad + \int_u^1\int_0^1 \left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}} \left(\frac{st}{(s+t)^2}\right)^{\frac{b}{2}} (s+t)^{\frac{\alpha_1}{2}} s^{\frac{\alpha_2}{2}} t^{\frac{\beta}{2}} \frac{ds}{s}\frac{dt}{t}\Bigg) \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \, u^{\frac{\alpha_1+\alpha_2+\beta}{2}},
\end{aligned}$$

using that $\alpha_1 \in (0,1), P^2 \in \mathsf{StGC}^0$ and $(\alpha_1 + \alpha_2 + \beta) > 0$.

For the term B, we have

$$\begin{aligned}
|B| &\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \int_{x',x''}\int_0^u K_{Q_u}(x,x') K_{P_t^{1\bullet}}(x',x'') d(x',x'')^{\alpha_1} t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}\nu(dx')\nu(dx'') \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \int_0^u t^{\frac{\alpha_1+\alpha_2+\beta}{2}}\frac{dt}{t} \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \, u^{\frac{\alpha_1+\alpha_2+\beta}{2}},
\end{aligned}$$

using again that $\alpha_1 \in (0,1)$ and $(\alpha_1 + \alpha_2 + \beta) > 0$.

Finally for C, we also use cancellation properties to get

$$
\begin{aligned}
|C| &\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\bigg\{\int_{x',x''}\int_u^1 K_{Q_u}(x,x')K_{P_t^1\bullet}(x',x'')\Big|a_1(x)-a_1(x')\Big|t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}\nu(dx')\nu(dx'') \\
&\quad+\int_{x',x''}\int_u^1 K_{Q_u}(x,x')K_{P_t^1\bullet}(x',x'')\Big|a_1(x')-a_1(x'')\Big|t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}\nu(dx')\nu(dx'')\bigg\} \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\bigg\{\int_{x',x''}\int_u^1 K_{Q_u}(x,x')K_{P_t^1\bullet}(x',x'')d(x,x')^{\alpha_1}t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}\nu(dx')\nu(dx'') \\
&\quad+\int_{x',x''}\int_u^1 K_{Q_u}(x,x')K_{P_t^1\bullet}(x',x'')d(x',x'')^{\alpha_1}t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}\nu(dx')\nu(dx'')\bigg\} \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\bigg\{u^{\frac{\alpha_1}{2}}\int_u^1 t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t}+\int_u^1\left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}}t^{\frac{\alpha_1+\alpha_2+\beta}{2}}\frac{dt}{t}\bigg\} \\
&\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta}{2}},
\end{aligned}
$$

using that $\alpha_1\in(0,1)$ and $(\alpha_2+\beta)<0$. In the end, we have

$$
\Big\|Q_u\mathsf{C}(a_1,a_2,b)\Big\|_\infty \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta}{2}}
$$

uniformly in $u\in(0,1]$, so the proof is complete for C. The adaptation of the proof to the case $a_1\in\mathcal{H}^{\alpha_1}$ is left to the reader and follows from the estimates of the Appendix A.1.

□

As explained, one needs refined correctors to gain from Hölder regularity greater than one. In the following Theorem, we prove the needed estimates for the first refined corrector which is the only one needed for (gPAM) equation in dimension 3 and (gKPZ) equation in dimension $1+1$. Recall that it is given for any $x\in M$ by

$$
\mathsf{C}_{(1)}\big(a,b,c\big)(x) := \mathsf{C}\big(a,b,c\big)(x)-\sum_{i=1}^{\ell}\gamma_i\big(V_ia\big)(x)\mathsf{\Pi}\big(\widetilde{\mathsf{P}}_{\delta_i(x,\cdot)}b,c\big)(x)
$$

where δ_i is given for $x,y\in M$ by

$$
\delta_i(x,y) := \chi\big(d(x,y)\big)\langle V_i(x),\pi_{x,y}\rangle_{T_xM}
$$

with χ a smooth non-negative function on $[0,+\infty)$ equal to 1 in a neighbourhood of 0 with $\chi(r)=0$ for $r\geq r_m$ the injectivity radius of the compact manifold M and $\pi_{x,y}$ a tangent vector of T_xM of length $d(x,y)$, whose associated geodesic reaches y at time 1. The functions γ_i are defined from the identity

$$
\nabla f = \sum_{i=1}^{\ell}\gamma_i(V_if)V_i,
$$

for all smooth real-valued functions f on M.

Theorem A.9. *Let $\alpha_1\in(1,2)$ and $\alpha_2,\beta\in\mathbb{R}$. If*

$$
\alpha_2+\beta<0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta>0
$$

then the operator $\mathsf{C}_{(1)}$ has a natural extension as a continuous operator from $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta}$.

Proof : For the continuity estimate of $\mathsf{C}_{(1)}$, we also want to compute the regularity using a family Q of StGC^r with $r > |\alpha_1 + \alpha_2 + \beta|$. Again a term $\Pi(a,b)$ can be written as a linear combination of terms of the form

$$\int_0^1 P_t^{1\bullet}\big(Q_t^1 a \cdot Q_t^2 b\big)\frac{\mathrm{d}t}{t},$$

while $\widetilde{\mathsf{P}}_b a$ is a linear combination of terms of the form

$$\int_0^1 \widetilde{Q}_t^{3\bullet}(\widetilde{Q}_t^4 a \cdot P_t^2 b)\frac{\mathrm{d}t}{t},$$

with $Q^1, Q^2, \widetilde{Q}^3, \widetilde{Q}^4 \in \mathsf{StGC}^{\frac{b}{4}}$ and $P^1, P^2 \in \mathsf{StGC}^{[0,b]}$. For the terms where $P^2 \in \mathsf{StGC}^{[2,3]}$, we already have the correct regularity since

$$\begin{aligned}\int_0^1\int_0^1 Q_u P_t^{1\bullet}\Big(Q_t^1\widetilde{Q}_s^{3\bullet}\Big(P_s^2 a_1 \cdot \widetilde{Q}_s^4 a_2\Big)\cdot Q_t^2 b\Big)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}& \\ \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta \int_0^1\int_0^1 &\left(\frac{ut}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{ts}{(s+t)^2}\right)^{\frac{3}{2}} s^{\frac{\alpha_1+\alpha_2}{2}} t^{\frac{\beta}{2}}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\ \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\, u^{\frac{\alpha_1+\alpha_2+\beta}{2}}&\end{aligned}$$

using that $\alpha_1 \in (1,2)$ so we only consider $P^2 \in \mathsf{StGC}^{[0,1]}$. For $P^2 \in \mathsf{StGC}^0$, we control it using the term $a_1\Pi(a_2,b)$ as in the proof of the continuity estimate of C. We are left with

$$\int P_t^{1\bullet}\left(Q_t^1\widetilde{Q}_s^{3\bullet}\Big(P_s^2\big(a_1 - \sum_{i=1}^{\ell}\gamma_i(V_i a_1)(x)\delta_i(\cdot,x)\big)\cdot\widetilde{Q}_s^4 a_2\Big)\cdot Q_t^2 b\right)(x)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}$$

with $P^2 \in \mathsf{StGC}^1$. Then the result follows with the same proof using that $P_s^2 1 = 0$ since it encodes some cancellation and the first order Taylor expansion

$$\left|a_1(y) - a_1(x) - \sum_{i=1}^{\ell}\gamma_i(V_i a_1)(x)\delta_i(y,x)\right| \lesssim d(x,y)^\alpha.$$

$\square$

We end this Section with the estimate on the commutator of the divergence. Recall the definition

$$\mathsf{B}\big(a,(b_1,b_2)\big) = \mathrm{div}\big(\mathsf{P}_a(b_1,b_2)\big) - \mathsf{P}_a\mathrm{div}(b_1,b_2).$$

Proposition A.10. *Let $\alpha < 1$ and $\beta \in \mathbb{R}$. Then $(a,b) \mapsto \mathsf{B}(a,b)$ extends in a unique bilinear operator from $\mathcal{H}^\alpha(M,\mathbb{R}) \times \mathcal{C}^\beta(\mathbb{R}^2,\mathbb{R}^2)$ to $\mathcal{H}^{\alpha+\beta-1}(M,\mathbb{R})$.*

Proof : We have

$$\mathsf{B}\big(a,(b_1,b_2)\big) = \partial_1\mathsf{P}_a b_1 + \partial_2\mathsf{P}_a b_2 - \mathsf{P}_a\partial_1 b_1 + \mathsf{P}_a\partial_2 b_2$$

hence the result follows from the continuity estimates on the commutators

$$(a,b) \mapsto V_i\mathsf{P}_a b - \mathsf{P}_a V_i b$$

proved in the following Section in the context of parabolic Hölder spaces. The extension to Sobolev spaces can be done as previously.

$\square$

A.3 – Additional correctors and commutators

The proofs of the corrector estimates follow the line of reasoning of similar estimates proved in the previous Section. We do not details the proof, see [11] for the details. As in Chapters 2 and 3, we denote here by $\mathcal{C}$ the parabolic Hölder spaces and the standard families of operators are also taken in spacetime. Recall the definitions of the following operators

$$
\begin{aligned}
\mathsf{C}_L^{<}\big(a_1, a_2, b\big) &= \mathsf{P}_{L\widetilde{\mathsf{P}}_{a_1 a_2}} b - a_1 \mathsf{P}_{La_2} b, \\
\mathsf{C}_L^{>}\big(a, b_1, b_1\big) &= \mathsf{P}_{La}\widetilde{\mathsf{P}}_{b_1} b_2 - b_1 \mathsf{P}_{La} b_2, \\
\mathsf{C}_L\big(a_1, a_2, b\big) &= \Pi\big(L\widetilde{\mathsf{P}}_{a_1} a_2, b\big) - a_1 \Pi\big(La_2, b\big).
\end{aligned}
$$

Theorem A.11.

- *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta - 2 < 0 \quad \textit{and} \quad \alpha_1 + \alpha_2 + \beta - 2 > 0$$

then the operators $\mathsf{C}_L^{<}$ and C_L have natural extensions as continuous operators from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.

- *Let $\beta_1 \in (0,1)$ and $\alpha, \beta_2 \in \mathbb{R}$. If*

$$\alpha + \beta_2 - 2 < 0 \quad \textit{and} \quad \alpha + \beta_1 + \beta_2 - 2 > 0$$

then the operator $\mathsf{C}_L^{>}$ has a natural extension as a continuous operator from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta_1} \times \mathcal{C}^{\beta_2}$ to $\mathcal{C}^{\alpha+\beta_1+\beta_2-2}$.

- *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta - 1 < 0 \quad \textit{and} \quad \alpha_1 + \alpha_2 + \beta - 1 > 0$$

then the operators $\mathsf{C}_{V_i}^{<}$ and C_{V_i} have natural extensions as continuous operators from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-1}$.

- *Let $\beta_1 \in (0,1)$ and $\alpha, \beta_2 \in \mathbb{R}$. If*

$$\alpha + \beta_2 - 1 < 0 \quad \textit{and} \quad \alpha + \beta_1 + \beta_2 - 1 > 0$$

then the operator $\mathsf{C}_{V_i}^{>}$ has a natural extension as a continuous operator from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta_1} \times \mathcal{C}^{\beta_2}$ to $\mathcal{C}^{\alpha+\beta_1+\beta_2-1}$.

As one needs refined corrector to gain from regularity exponent greater than one, the same is true for the modified correctors, this is the following Theorem for $\mathsf{C}_{L,(1)}$.

Theorem A.12.

- *Let $\alpha_1 \in (1,2)$ and $\alpha_2, \beta \in \mathbb{R}$. If*

$$\alpha_2 + \beta - 2 < 0 \quad \textit{and} \quad \alpha_1 + \alpha_2 + \beta - 2 > 0$$

then the operators $\mathsf{C}_{L,(1)}^{<}$ and $\mathsf{C}_{L,(1)}$ have natural extension as continuous operators from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.

- *Let $\beta_1 \in (1,2)$ and $\alpha, \beta_2 \in \mathbb{R}$. If*

$$\alpha + \beta_2 - 2 < 0 \quad \textit{and} \quad \alpha + \beta_1 + \beta_2 - 2 > 0$$

then the operator $\mathsf{C}_{L,(1)}^{>}$ has a natural extension as a continuous operator from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta_1} \times \mathcal{C}^{\beta_2}$ to $\mathcal{C}^{\alpha+\beta_1+\beta_2-2}$.

The continuity results on R_V and R_L are obtained as for R. This also follows from the estimate on V and L from [11] recalled here.

Theorem A.13.

- *Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$. Then the operator L has a natural extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ into $\mathcal{C}^{\alpha+\beta-2}$.*

- *Let $\alpha_1, \alpha_2 \in (0,1)$ and $\beta \in \mathbb{R}$. Then the iterated operator*

$$\mathsf{L}\big((a_1, a_2), b\big) := \mathsf{L}\big(\mathsf{P}_{a_1} a_2, b\big) - \mathsf{P}_{a_1}\mathsf{L}(a_2, b)$$

  has a natural extension as a continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^\beta$ into $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.

- *Let $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$ and $\beta \in \mathbb{R}$. Then the iterated operator*

$$\mathsf{L}\Big(\big((a_1, a_2), a_3\big), b\Big) := \mathsf{L}\big((\mathsf{P}_{a_1} a_2, a_3), b\big) - \mathsf{P}_{a_1}\mathsf{L}\big((a_2, a_3), b\big)$$

  has a natural extension as a continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\alpha_3} \times \mathcal{C}^\beta$ into $\mathcal{C}^{\alpha_1+\alpha_2+\alpha_3+\beta-2}$.

- *Let $\alpha \in (1,2)$ and $\beta \in \mathbb{R}$. Then the operator $\mathsf{L}_{(1)}$ has a natural extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ into $\mathcal{C}^{\alpha+\beta-2}$.*

- *Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$. Then the operator V_i has a natural extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta-1}$.*

- *Let $\alpha_1, \alpha_2 \in (0,1)$ and $\beta \in \mathbb{R}$. Then the iterated operator*

$$\mathsf{V}_i((a_1, a_2), b) := \mathsf{V}_i(\mathsf{P}_{a_1} a_2, b) - \mathsf{P}_{a_1}\mathsf{V}_i(a_2, b)$$

  has a natural extension as a continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-1}$.

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Titre : Un calcul paracontrôlé pour les EDP stochastiques singulières sur les variétés : vers l'infini et au-delà

Mots clés : EDP stochastiques singulières, renormalisation, calcul paracontrôlé

Résumé : Cette thèse porte sur le calcul paracontrôlé construit à l'aide du semi-groupe de la chaleur pour étudier différentes équations aux dérivées partielles stochastiques singulières sur des variétés riemanniennes compactes. En utilisant la formule de Calderón comme analogue continu à la décomposition de Paley-Littlewood, on peut construire un paraproduit dans un tel cadre géométrique. Il est alors possible de donner un sens à une large classe d'équations paraboliques semi-linéaires incluant les équations généralisées de KPZ en dimension 1+1 et du modèle parabolique d'Anderson en dimension 3. On montre ensuite que cette méthode peut être étendue pour la résolution des versions quasi-linéaires de ces équations en adaptant simplement les outils du calcul paracontrôlé et en généralisant la notion de système paracontrôlé à des familles infinies générées par une structure algébrique finie.

Un autre problème abordé dans cette thèse est l'étude des opérateurs stochastiques singuliers comme l'hamiltonien d'Anderson, c'est-à-dire l'opérateur de Schrödinger avec comme potentiel un bruit blanc espace. Après une étape de renormalisation, le calcul paracontrôlé permet la définition de cet objet en tant qu'opérateur auto-adjoint à spectre discret. D'autres opérateurs sont aussi étudiés comme le laplacien magnétique avec un champ magnétique bruit blanc. L'étude de ce type d'opérateur permet la résolution d'équations d'évolutions associées ou l'étude de modèles aléatoires continus. On obtient ainsi des inégalités de Strichartz pour les équations de Schrödinger et des ondes avec un potentiel bruit blanc sur une surface compacte avec ou sans bord et on étudie les modèles de la mesure polymère en dimension 2 avec potentiel bruit blanc et la diffusion de Brox sur le cercle.

Title : A paracontrolled calculus for singular stochastic PDEs on manifolds : to infinity and beyond

Keywords : Singular stochastic PDEs, renormalisation, paracontrolled calculus

Abstract : This thesis presents the paracontrolled calculus built with the heat semigroup to study different singular stochastic partial differential equations on compact Riemannian manifolds. Using the Calderón formula as a continuous analogue of the Paley-Littlewood decomposition, we can construct a paraproduct in such geometrical framework. It is then possible to give a sense to a large class of semilinear parabolic equations including the generalised KPZ equation in dimension 1+1 and the generalised parabolic Anderson model equation in dimension 3. We show that this method can be extended to deal with the quasilinear version of these equations by a simple adaptation of the paracontrolled calculus and by generalizing the notion of paracontrolled systems to infinite families generated by a finite algebraic structure.

Another problem tackled in this thesis is the study of singular stochastic operators such as the Anderson Hamiltonian, that is the Schrödinger operator with as potential a space white noise. After a renormalisation step, paracontrolled calculus allows the definition of this object as a self-adjoint operator with discrete spectrum. Other operators are also studied such as the magnetic Laplacian with white noise magnetic field. Study of this type of operators allows the resolution of associated evolution equations or the study of continuous random models. We thereby obtain Strichartz inequalities for the Schrödinger and wave equations with a white noise potential on compact surfaces with or without boundary and we study the models of the polymer measure in two dimensions with white noise potential and the Brox diffusion on the circle.